TRAINING COURSE

ON

RESERVOIR OPERATION

(UNDER WORLD BANK AIDED HYDROLOGY PROJECT)

Module 8

Introduction

to

Linear Programming

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INTRODUCTION TO LINEAR PROGRAMMING

1.0 INTRODUCTION

Optimization is the science of choosing the best amongst a number of possible alternatives. For many engineering problems, a number of feasible solutions are available. It is required to evaluate each alternative and then choose the best from the point of view of interest, say economic or convenience etc. The best solution under the given circumstances is known as the optimum solution.

The variables which are unknown, and are manipulated to obtain the optimum solution are termed as decision variables. The decision maker evaluates the available alternatives on the basis of some prescribed criterion. This criterion function is normally referred to as the objective function. The availability of resources is usually limited, and is expressed with the help of constraints. These constraints are imposed during the decision making process to determine the optimum solution.

The optimization techniques are also known as mathematical programming techniques. It can be classified in several ways, such as on the basis of the existence of constraints, the nature of design variables, the physical structure of the problem, the nature of the equations involved, the permissible values of the design variables, the deterministic nature of variables involved, the separability of the functions, the number of objective functions involved, the nature of the problem, etc. the usual way of classifying the optimization techniques is based on the nature of the problem or equations involved. Following this way, the techniques can be classified as Linear Programming (LP), Nonlinear Programming (NLP), Geometric Programming (GM), quadratic Programming (QP), and Dynamic Programming (DP). This classification is extremely useful from the computational point of view since there are many methods developed solely for the efficient solution of a particular class of problems. In addition, some other classes of optimization techniques also exist which are very much relevant for the hydrologists. These techniques are based on the statistical methods and stochastic process techniques. Now a days, many new types of optimization techniques are also emerging, such as Genetic Algorithm (GA), Simulated Annealing Algorithm (SAA), etc. However, one should not forget the classical methods of optimization which are analytical and make use of the techniques of differential calculus in locating the optimum points of continuous and differentiable functions. Moreover, the calculus methods of optimization form a basis for developing most of the numerical techniques of optimization.

Although the optimization encompasses a very wide range of subjects, keeping the objectives of this course and the current status of the application of optimization techniques in operational hydrology in our country, these notes is confined to the linear programming only. Some advanced topics in linear programming, such as duality and post optimality analysis, too are discussed briefly. After reading this course material, one would be able to understand the fundamentals of linear programming and the simplex method and get the idea of the potential use and worth of the optimization techniques in handling large and complex real-life hydrological problems.

2.0 LINEAR PROGRAMMING

The linear programming type of optimization problem was first recognized in the 1930s by economists while developing methods for the optimal allocation of resources. During World War II, the United States Air Force sought more effective procedures of allocating resources, and that lead to the development of the linear programming. G.B. Dantzig, who was a member of the Air Force Group, formulated the general linear programming problem, and devised the simplex method of solution in 1947. This has become a significant step in bringing the linear programming into wider usage. Since then, linear programming models are being used widely to solve a variety of military, economic, industrial, social, engineering and hydrological problems. The number of applications of linear programming has become so immense that it is not possible to describe all of them here.

The optimization problems in which the objective function and constraints are linear function of decision variables along with the condition that the decision variables are positive are termed as linear programming problems. Thus, any optimization problem can be classified as a LP problem provided it meets the following conditions:

○   The decision variables involved in the problem are nonnegative, i.e., positive or zero.

○   The criterion function or objective function is described by a linear function of the decision variables, that is, a mathematical function involving only the first powers of the variables with no cross products.

○   The operating rules governing the processes, commonly known as constraints, are expressed as a set of linear equations or linear inequalities.

The LP models have been extensively used to solve different kinds of small to large water resources problems. Although the objective function and the constraints are not related linearly with the decision variables in many real-life water resources problems, these can be approximated by linear functions, and LP technique can be used to obtain the solution.

3.0 MATHEMATICAL REPRESENTATION OF A LP PROBLEM

There are many ways to represent a linear programming problem. A general linear program formulated for a real-life or hypothetical problem will look like a conventional type. Solutions are generally sought by converting the general linear program to a standard form. A compact and computationally desirable way of representation of a linear program is a matrix form. Each form is described in following section.

3.1 Conventional Form

A general linear program consists of a linear objective function and a set of linear constraints. The constraints may be expressed in terms of inequalities or equalities. In most cases, especially in

real-life water resources problems, constraints appear as inequalities. In many cases, it is observed that a linear program may consist of both types of constraints, i.e., some constraints may be equality type and some may be inequality type constraints.

Mathematical representation of a general linear program is often referred to as the conventional form. A general linear programming problem in conventional form can be written as:

Minimize (or Maximize): $Z = f(x)$ ...(1)

Subject to:

$$g_j(x) \leq 0; \qquad j=1, 2, \ldots, m_1 \qquad \text{...(2)}$$
$$h_j(x) \geq 0; \qquad j=1, 2, \ldots, m_2 \qquad \text{...(3)}$$
$$l_j(x) = 0; \qquad j=1, 2, \ldots, m_3 \qquad \text{...(4)}$$

where Z is objective function, x is an n-dimensional decision vector, $g_j(x)$ and $h_j(x)$ are inequality constraints, and $l_j(x)$ is equality constraint. m_1, m_2 and m_3 denote the number of constraints for these types of constraints, respectively. The objective function is expressed as a linear function of the decision variables.

3.2 Standard Form

The standard form of a linear programming problem with m constraints and n variables can be represented as follows:

Minimize (or Maximize): $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ ...(5)

Subject to:

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$
$$a_{22} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_n$$
$$\begin{matrix} . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \end{matrix} \qquad \text{...(6)}$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m$$

$$x_i \geq 0, \quad i = 1, 2, \ldots n. \qquad \text{...(7)}$$
$$b_i \geq 0, \quad i = 1, 2, \ldots m. \qquad \text{...(8)}$$

where Z represents the objective function; x_i are the decision variables and c_i are the cost (or benefit) coefficients representing the cost (or benefit) incurred by increasing the x_i decision variable by one unit. The right hand side of constraint equations represents the resource availability. These arise due to limited availability of a particular resource, say water. The a_{ij} coefficients are called technological coefficients and quantify the amount of a particular resource i

required per unit of the activity j.

The standard form of a LP problem is useful for solving the LP model algebraically. The main features of the standard form are:

o   The objective function is of the either maximization or minimization type,

o   All the constraints are expressed as equations, i.e. equality type constraints except the nonnegativity constraints associated with the decision variables,

o   All the decision variables are restricted to be nonnegative,

o   The RHS constant of each constraint is nonnegative.

3.3 Matrix Form

In matrix notation, the standard linear programming problem can be expressed in a compact form as:

$$\text{Min. (or Max.)} \quad Z = C^T X \quad \quad \ldots(9)$$

Subject to:
$$A X = b \quad \ldots(10)$$
$$X \geq 0 \quad \ldots(11)$$
$$b \geq 0 \quad \ldots(12)$$

where A is an (m x n) matrix, X is an (n x 1) column vector, b is an (m x 1) column vector, C is a (n x 1) column vector, and Z represents the objective function. The terms "Min." and "Max." refer to minimization and maximization, respectively, and notation T refers to the transpose operation. In other words, it can be expressed as:

$$X = [x_1, \ x_2, \ \ldots\ldots , \ x_n]^T \quad \ldots(13)$$
$$C = [c_1, \ c_2, \ \ldots\ldots , \ c_n]^T \quad \ldots(14)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \ldots(15)$$

$$b = [b_1, \ b_2, \ \ldots\ldots , \ b_m]^T \quad \ldots(16)$$

4.0 FORMULATION OF A LP MODEL

The three basic steps in constructing a linear programming model are as follows:

o   Identify the decision variables to be determined, and represent them in terms of algebraic symbols.

o   Identify all the restrictions or constraints in the problem, and express them as linear equations or inequalities which are linear functions of the decision variables.

o   Identify the objective or criterion, and represent it as a linear function of the decision variables, which is to be either maximized or minimized.

These three basic steps will be more clear when one formulates a number of linear programs. Here, one should notice that the model building is not a science, but primarily an art and comes mainly by practice. Depending upon the experience, skill and scientific knowledge about the system under consideration, the developed model will meet the realism, and will fulfil the intended objectives. Any discrepancy in the model formulation will yield an erroneous result, and sometimes may even show physically meaningless solution. Hence, the readers are advised to work out many exercises on problem formulation before handling a real life problem of public welfare or direct implementation.

5.0 ASSUMPTIONS IN LP

There are four basic assumptions which are implicitly built into LP models:

a) Proportionality Assumption

It is assumed that the contribution of the jth decision variable to the effectiveness measure, $c_j x_j$, and its usage of the various resources, $a_{ij} x_j$, are directly proportional to the decision variable.

b) Additivity Assumption

This assumption indicates that the total usage of resources and contribution to the overall measure of effectiveness are equal to the sum of the corresponding quantities generated by each activity conducted by itself at the given level.

(c) Divisibility Assumption

According to this assumption, the fractional values of the decision variables are permissible.

(d) Deterministic Assumption

The parameters of the LP model are assumed to be known with certainty.

6.0 REDUCTION OF A GENERAL LINEAR PROGRAM TO A STANDARD FORM

The simplex method for solving linear programming problems requires problem to be expressed in standard form. But not all linear programming problems appear like standard form. In many cases, some of the constraints are expressed as inequalities rather than equations, it is most often true in case of water resources problems. In some problems, all the decision variables may not be even nonnegative. Hence the first step in solving a linear program is to convert.it to a problem in standard form. The procedure to convert a general program to a linear program is outlined below:

o   Convert all inequalities to equalities.
o   Convert all decision variables unrestricted in sign to strictly non-negative.
o   Make all the right-hand side constants of the constraints nonnegative.

6.1 Handling Inequality Constraints

An inequality constraint of the type $\leq$ can be converted to equality type by introducing a new nonnegative variable called a slack variable. This new variable is added in the left-hand side of the constraint. Hence, the constraint

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \qquad \ldots(17)$$

can be written as:

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + s_1 = b_1; \quad s_1 \geq 0 \qquad \ldots(18)$$

Here s_1 represents the slack variable.

Similarly, an inequality constraint of the type $\geq$ can be converted to equality type by introducing a new nonnegative variable called a surplus variable. This new variable is subtracted from the left-hand side of the constraint. Thus, the constraint

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \geq b_1 \qquad \ldots(19)$$

can be written as:

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n - s_2 = b_1; \qquad s_2 \geq 0 \qquad \ldots(20)$$

Here s_2 represents the surplus variable. Further, a constraint of $\geq$ type, if desired, can be easily converted to a $\leq$ type by multiplying by -1 throughout the equation.

6.2 Handling Variables Unrestricted in Sign

In some situations, it may become necessary to introduce a variable in the optimization model that can be both positive and negative values. Since the standard form requires all the variables to be nonnegative, an unrestricted variable must be transformed. For this, the unrestricted variable is replaced by the difference of the two nonnegative variables.

6.3 Handling Constraints Having Negative Right-Hand Side Constants

Since the right-hand side constant of each constraint must be nonnegative, the constraints having negative right-hand side constants are multiplied by -1 throughout to get the constraint in the standard form. Thus, the constraint

$$3x_1 - x_2 - 2x_3 = -5 \qquad \qquad ...(21)$$

will take the form

$$-3x_1 + x_2 + 2x_3 = 5 \qquad \qquad ...(22)$$

in standard form. It should be noted here that if inequality type constraints are being multiplied by -1, their nature will reverse.

6.4 Interchanging the Nature of the Objective Function

The nature of the objective function, if desired, can be changed by putting a negative sign with the prescribed expression for the objective function. That means a maximization problem is equivalent to a minimization problem with the negative of the objective function, i.e.,

$$\text{Max } [Z] = \text{Min } [-Z] \qquad \qquad ...(23)$$

7.0 CANONICAL SYSTEM

A system of equations which possesses at least one basic variable in all equations is called a canonical system. In order to get a canonical system, a sequence of pivot operations are performed on the original system such that there exists at least one basic variable in each equation. The number of basic variables is decided by the number of equations in the system.

A variable is said to be a basic variable in a given equation if it appears with a unit coefficient in that equation and zeros in all other equations. Those variables which are not basic are called nonbasic variables. By applying the elementary row operations, a given variable can be made a basic variable.

A system of equations given by

$$x_1 - 3x_3 - 2x_4 - 4x_5 = 6 \qquad \qquad ...(24)$$
$$x_2 - 2x_3 + x_4 - 3x_5 = 2 \qquad \qquad ...(25)$$

represents a canonical system. This system is very useful to obtain the optimal solution, and forms a basis for the simplex method to find out an optimal solution of an optimization problem.

The solution obtained from a canonical system by setting the nonbasic variables equal to zero and solving for the basic variables is called a basic solution. A basic feasible solution is a basic solution in which the values of the basic variables are nonnegative. The basic feasible solution satisfies

all constraints. A basic feasible solution which provides minimum (or maximum) value of the objective function is called an optimum solution. The feasible region and constraints are shown in Fig. 1. It may be noted that the feasible region of a properly formed LP problem is a convex set. A set is a convex set if it is not possible to find two points such that not all points on the line joining them belong to the set.
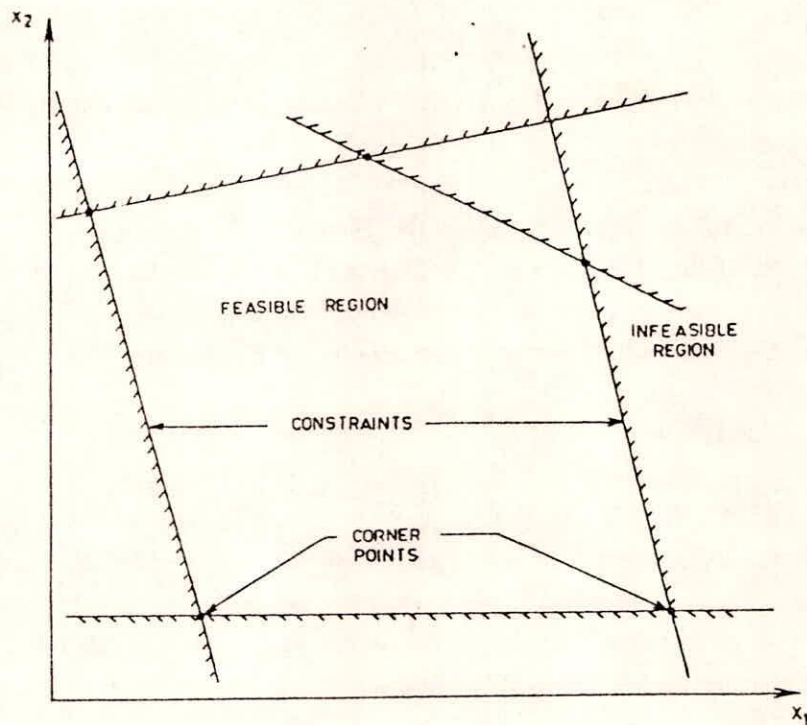

Fig. 1 A Definition Sketch of Feasible Region and Constraints

8.0 GRAPHICAL SOLUTION

The graphical method is a simple way to solve LP problems. This method is also very useful in conceptual understanding of the solution technique. However, it can be used to solve the LP problems involving at most two decision variables only. In the following, a LP problem having two decision variables will be discussed.

8.1 Illustrative Example 1

$$Max \ Z = 2 x_1 + x_2 \qquad \qquad ...(26)$$

subject to:

$$2x_1 - x_2 \leq 8 \qquad \qquad ...(27)$$
$$x_1 + x_2 \leq 10 \qquad \qquad ...(28)$$
$$x_2 \leq 7 \qquad \qquad ...(29)$$
$$x_1, x_2 \geq 0 \qquad \qquad ...(30)$$

In Fig. 2, the constraints are plotted against the coordinate axes x_1 and x_2. To plot the first constraint, $2x_1 - x_2 \leq 8$, we plot a straight line $2x_1 - x_2 = 8$. Similarly, we plot lines $x_1 + x_2 = 10$, and $x_2 = 7$ to mark second and third constraints. The nonnegativity constraints are plotted as the axes themselves. The feasible region can be easily delineated, and is shown in Fig. 2 by the bounded pentagon region formed by the hatched lines of each constraints including nonnegativity. Now we start with an arbitrary value of objective function, say 6 and plot the line $2x_1 + x_2 = 6$. Since it is a maximization problem, the objective function line is shifted forward as far as possible while ensuring that at least one point lies in the feasible region. It can be seen that the farthest point up to which we can go is the point (6, 4). Hence, this is the optimum point at which the objective function becomes maximum. Thus, the optimal solution of the problem is $x_1 = 6$ and $x_2 = 4$, and the optimal value of the objective function is 16.
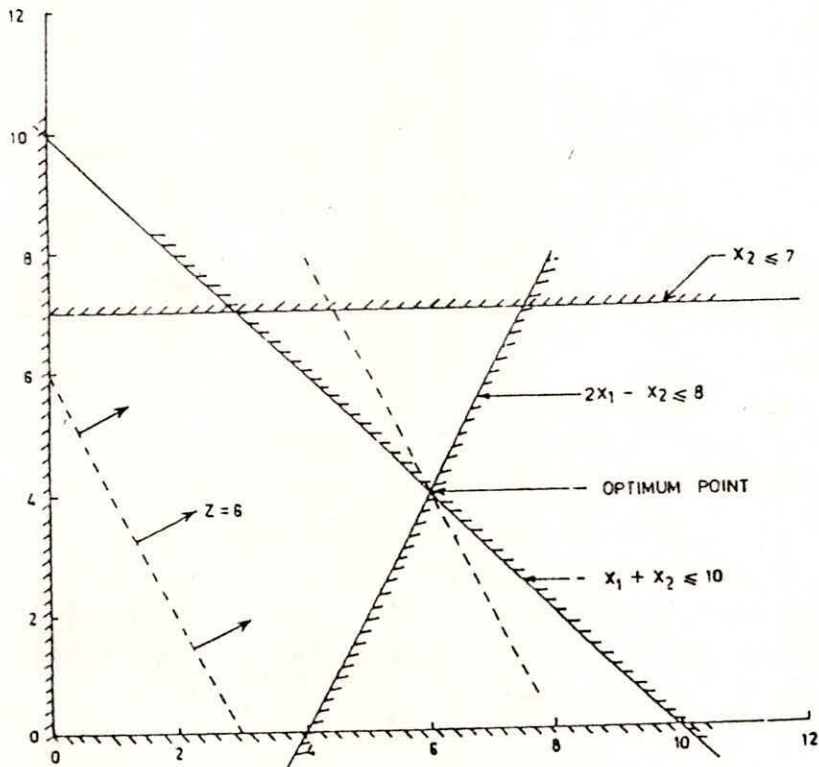

Fig. 2 Graphical Solution of Illustrative Example 1

8.2 Importance of Corner Points

A closer inspection of Fig. 2 shows that the optimum point will always be a corner point. Further, if a constraint passing through this corner point is parallel to the objective function, all the points falling on this constraint (or objective function) will have the same (optimum) value of the objective function. In this case, the problem will have infinite solutions. Such linear programming problems are said to have alternative or multiple optimal solutions. Some important properties of the corner points are described below:

○   If there is only one optimal solution to a LP problem, it must be a feasible extreme point. If

there are multiple optimal solutions, at least two must be adjacent feasible extreme points.

o   In every problem, there are only a finite number of feasible extreme points.

o   If a feasible extreme point is better than all its adjacent feasible points, then it is better than all other feasible extreme points. This property holds good if the feasible region is convex. Based on this property, one need not enumerate all the extreme points, and the status of one extreme point can be ascertained to determine whether optimal solution has been arrived at or not.

In some problems the feasible region may not be a closed convex polygon, and hence it may be possible to increase the objective function value continuously and still be inside this region. Such type of problems are termed as having unbounded solution, and no finite optimum could be achieved in such cases.

9.0 SIMPLEX METHOD

The graphical method of solving a LP problem is suitable only for problems involving two or less decision variables. As the size of the problem increases, this method cannot be used, and a method called Simplex Method is used to obtain an optimal solution.

In case of any formulated optimization problem, one of the three cases may arise. These three cases are: (i) $m = n$, (ii) $m > n$, and (ii) $m < n$. If the number of decision variables n is equal to the number of constraint equations m, the problem has a unique solution, if it exists, and it is said that there can be no optimization in this case. If $m > n$, then there would be $m-n$ redundant equations which could be eliminated. If $m-n$ equations are not redundant, it has a solution only in least square sense. The case $m < n$ corresponds to an undetermined set of linear equations which, if they have a solution, will have an infinite number of solutions. The problem of linear programming is to find one of these solutions which satisfies our constraints and yield the optimum value of the objective function. In case of $m < n$, we can set $(n-m)$ variables equal to zero and solve the m equations for m variables. These solutions will be basic solutions as $(n-m)$ variables which have been set equal to zero represent nonbasic variables. However, there will be ^nC_m such solutions.

If in a particular problem, the number of decision variables, $n = 20$, and the number of constraints, $m = 10$, then the number of possible basic solutions will be $^{20}C_{10} = 20!/[(20-10)!10!]$ $= 184756$. Hence to solve this problem, 184756 solutions will be required to be obtained and compared. This is formidable task even with the help of a fast digital computer.

A very efficient method was developed by Dantzig which is called the Simplex Method. The simplex method is an efficient iterative procedure for solving linear programming problems expressed in standard form. In addition to the standard form, the simplex method requires that the constraint equations be expressed as a canonical system from which a basic feasible solution can be readily obtained. The simplex method begins with this basic feasible solution. Once a basic feasible solution is available, attempt is made to improve it till the optimal solution is not obtained.

The equations containing only the slack variables can be automatically considered as canonical system and these slack variables as basic variables. However, in many cases, it is observed that finding a canonical system with a basic feasible solution is not an easy task. One way to obtain a basic feasible solution is to arbitrarily choose the basic variables and use a technique like Gauss Elimination to get the solution. Another approach, but systematic, of getting a canonical system with a basic feasible solution is the use of artificial variables. The artificial variables are added in those equation in which no basic variables appear by inspection. An auxiliary objective function is formed which is equal to the sum of artificial variables. This method is called two phase simplex as we have two objective functions. The first phase aims at minimization of the auxiliary objective function. If as a result of this phase, this function can not be made zero then the problem is infeasible and the algorithm is terminated. If the auxiliary objective unction is zero, then the optimization of the main function is taken up.

9.1 Computational Steps of the Simplex Method

The computational steps of the simplex method in tableau form are as follows:

❶ Express the problem in a standard form.

❷ Start with an initial basic feasible solution in canonical form and set up the initial tableau.

❸ Use the inner product rule to find the relative profit (or cost) coefficients ($\overline{C}_j$). The inner product rule states that the relative profit (or cost) coefficient of the variable x_j ($\overline{C}_j$) is obtained by subtracting the product of the row matrix consisting of profit (or cost) coefficients (C_j) of basic variables and the column matrix consisting of transformation coefficients (a_{ij}) corresponding to x_j in the canonical system from the actual profit (or cost) coefficient corresponding to the variable x_j. If all the relative profit coefficients are negative or zero, the current basic feasible solution is optimal for a maximization problem. Otherwise, select the nonbasic variable with the most positive $\overline{C}_j$ value (highest value) to enter the basis. For a minimization problem, all $\overline{C}_j$ should be positive or zero at the optimal point. In this case, the nonbasic variable with the most negative $\overline{C}_j$ value (lowest value) is selected to enter the basis. The decision is arbitrary in case of a tie. Let this variable be x_r. The value of the objective function can be also computed by multiplying the row matrix consisting of profit (or cost) coefficients of basic variables and the column matrix consisting of right-hand side constants.

❹ Apply the minimum ratio rule to determine the basic variable to leave the basis. The minimum ratio rule is that for this variable (x_r), take (b_i/a_{ir}) ratio for each constraint row i (for those constraints only which have +ve a_{ir} values), and the minimum ratio determines the row in which the basic variable will have unit coefficient. The corresponding variable from this row (which was a basic variable) will leave the basis. The constraint row corresponding to the entering basic variable is known as pivot equation and the element located at the intersection of the entering column and pivoting row is known as the pivot element.

❺ Perform the pivot operation to get the new tableau in canonical form, and a new basic feasible solution is obtained from this form.

❻ Go to step ❸ and repeated the steps until an optimal solution is found.

If during the simplex iterations, the value of one or more basic variables become zero, it is termed as a degenerate solution. In such an event, there is no assurance that the solution will improve further. Sometimes, the degeneracy is temporary, and the solution improves after a few iterations.

The Simplex method, when used for a large problem requires considerable computer time and storage. Some techniques have been developed which require lesser time and storage. Among these techniques, the Revised Simplex method is the most popular in which the time and storage is saved by manipulating only selected entries of the Simplex Tableau.

9.2 Illustrative Example 2

The problem illustrated under section 8 (Example 1; Eqs. 26-30) is again solved here using the simplex method to demonstrate the step by step procedure of this method. Writing this problem in the canonical form by introducing slack variables, x_3, x_4 and x_5,:

$$\text{Max } Z = 2 x_1 + x_2 \qquad \qquad \text{...(31)}$$

subject to:

$$2x_1 - x_2 + x_3 \qquad = 8 \qquad \qquad \text{...(32)}$$
$$x_1 + x_2 \qquad + x_4 \qquad = 10 \qquad \qquad \text{...(33)}$$
$$x_2 \qquad + x_5 \qquad = 7 \qquad \qquad \text{...(34)}$$
$$x_1, x_2, x_3, x_4, x_5 \qquad \geq 0 \qquad \qquad \text{...(35)}$$

In this problem, the values of n and m are 5 and 3, respectively. Hence, there will be three basic variables which could be chosen arbitrarily. In this case, the variables x_3, x_4 and x_5 can be considered to be basic variables, and thus the problem is in the canonical form. This finishes Step **❶**, let us go to Step **❷**. The initial basic feasible solution will be $x_1 = 0$, $x_2 = 0$, $x_3 = 8$, $x_4 = 10$ and $x_5 = 7$. The initial Simplex Tableau is formed as follows:

Tableau 1

C_j	2	1	0	0	0	
	x_1	x_2	x_3	x_4	x_5	RHS
	2	-1	1	0	0	8
	1	1	0	1	0	10
	0	1	0	0	1	7
$\bar{C_j}$	2	1	0	0	0	Z = 0

Here, one should notice that the contents below the dotted line are not the part of the initial Simplex Tableau. The bottom row separated by the dotted line shows the value of the relative profit or cost coefficients corresponding to the variable x_j ($\bar{C_j}$). It is computed as stated in Step **❸**. Since, two values of relative profit coefficients are not either negative or zero, this is not the optimal

solution. We have to improve the basic feasible solution. For this, we see that the value of relative profit coefficient is maximum for x_1 variable, thus x_1 will enter the basis. This finishes the Step ❸. Now moving to Step ❹, since row 1 gives the minimum (b_i/a_{ir}) ratio, the variable x_3 will leave the basis. Thus, the row 1 is the pivot row and the number 2 in this row is the pivot element. This finishes Step ❹.

Coming to Step ❺, all the coefficients in the pivot row are divided by the pivot element, and x_r is eliminated from all rows except row 1. To make the coefficient of x_1 unity, row 1 is divided by 2. To eliminate x_1 from the second row, the row 1 is multiplied by -1/2 and is added to the second row. In row 3, already x_1 variable is not appearing. After performing these operations, a new Simplex Tableau is formed as follows:

Tableau 2

C_j	2	1	0	0	0	RHS
	x_1	x_2	x_3	x_4	x_5	
	1	-1/2	1/2	0	0	4
	0	3/2	-1/2	1	0	6
	0	1	0	0	1	7
C_j	0	2	-1	0	0	Z = 8

This new table shows the constraints in canonical form, and thus the improved basic feasible solution is $x_1 = 4$, $x_2 = 0$, $x_3 = 0$, $x_4 = 6$ and $x_5 = 7$. This finishes Step ❺, and now we move to Step ❻. Again, from the Table 2, it is clear that this solution is not an optimal solution because all C_j are not either negative or zero. Thus, we again repeat the process. A close inspection of Tableau 2 shows that variable x_2 will enter the basis and variable x_4 will leave the basis. Therefore, a new tableau is again formed with row 2 as the pivot row and the number 3/2 in this row as the pivot element. The new tableau is shown below:

Tableau 3

C_j	2	1	0	0	0	RHS
	x_1	x_2	x_3	x_4	x_5	
	1	0	1/3	1/3	0	6
	0	1	-1/3	2/3	0	4
	0	0	1/3	-2/3	1	3
C_j	0	0	-1/3	-4/3	0	Z = 16

Tableau 3 shows that all relative profit coefficients are either negative or zero. Thus the

optimal point has been reached and the computations are terminated. The optimal value of the objective function is 16 and the optimal solution is $x_1 = 6$, $x_2 = 4$, $x_3 = 0$, $x_4 = 0$ and $x_5 = 3$.

9.3 Interpreting Simplex Tableau

The final simplex tableau, besides giving information about the optimal solution, also contains other useful information. From the simplex tableau given above, one can readily determine that the values of basic variables are $x_1 = 6$ & $x_2 = 4$ and the value of the objective function is 16. The value of non-basic decision variables at the optimum point is zero except x_5. The values of optimum slack variables are ignored because they do not affect the decision to be taken. However, if a slack variable is a basic variable at optimal stage, the corresponding constraint is nonbinding and the corresponding resource is abundant. Otherwise, the constraint is binding and the resource is scarce. Note the RHS coefficients can be viewed as resource constraints, especially the $\leq$ type constraints.

The simplex tableau also contains information about the per unit worth of a resource, which is also known as its shadow price. This information is useful while fixing priorities about allocation of funds for various resources. The shadow price of a non-binding resources is zero while it is non-zero for a binding constraint. The per-unit worth of a resources is given by $\partial Z / \partial b_i$, $i = 1, 2, \ldots, m$. Any change in the availability of the resource corresponding to the binding constraints will change the optimum solution. The value of per-unit worth of a resource can be obtained from the final tableau in the objective function row under the starting basic feasible variables. In the example above, the per-unit worth of resources for the constraints no. 1, 2, and 3 are 1/3, 4/3, and 0, respectively. This implies that an increase of availability of resource 1 by one unit will lead to increase in objective function value by 1/3 units.

10.0 DUALITY IN LP

Associated with every LP problem (termed as the primal problem), there exists another problem known as the dual problem. The dual problem is formulated by transposing the rows and columns of the primal problem including the right-hand side and the objective function, reversing the inequalities and maximizing instead of minimizing.

Consider the following problem (Primal problem):

$$\text{Min } Z = C^T x \qquad \text{...(36)}$$

Subject to:

$$A x \geq b \qquad \text{...(37)}$$
$$x \geq 0 \qquad \text{...(38)}$$

Then the problem

$$\text{Max } Z' = b^T y \qquad \text{...(39)}$$

Subject to:

$$A^T y \leq C \qquad \text{...(40)}$$

$$y \geq 0 \qquad \qquad \ldots(41)$$

is termed as the Dual problem. Here y is a column vector (m x 1). To write a dual problem, it is necessary to write the primal problem in a particular way. In a minimization problem, all the constraints must be written in $\geq$ form and all the constraints of a maximization problem must be written in $\leq$ form. For example, let the primal be:

$$\text{Min} \quad z = x_1 + x_2 \qquad \qquad \ldots(42)$$
$$\text{ST} \quad x_1 + 2x_2 \geq 5 \qquad \qquad \ldots(43)$$
$$2x_1 + x_2 \geq 4 \qquad \qquad \ldots(44)$$
$$x_1, x_2 \geq 0 \qquad \qquad \ldots(45)$$

then the dual is

$$\text{Max} \quad z_1 = 5y_1 + 4y_2 \qquad \qquad \ldots(46)$$
$$\text{ST} \quad y_1 + 2y_2 \leq 1 \qquad \qquad \ldots(47)$$
$$2y_1 + y_2 \leq 1 \qquad \qquad \ldots(48)$$
$$y_1, y_2 \geq 0 \qquad \qquad \ldots(49)$$

10.1 Relationship Between Primal and Dual Problems

Some interesting relations exist between a primal problem and its dual. They are:

1. The dual of the dual is the primal.
2. If primal is a minimization problem then dual is a maximization problem.
3. If dual has a finite solution, then primal also has a finite solution.
4. For each variable in primal, there exists a constraint in the dual and vice versa.
5. If any variable in the primal is unrestricted in sign, then the corresponding constraint in the dual is an equality constraint and vice-versa.
6. If the primal has unbounded solution, the dual has either unbounded solution or is infeasible.
7. In the final solution (if it exists), if any constraint in the primal problem is satisfied as equality, the corresponding dual variable will have a value greater than zero, and vice versa.

The dual variables, y, are also termed as simplex multipliers, Lagrange multipliers, shadow prices, marginal costs or opportunity costs. If a constraint is viewed as a resources constraint, the dual variable gives the marginal value of relaxing the constraint. It shows the change in the objective function per unit change in the RHS at the optimum, all other things remaining same. Further, if the problem contains large number of constraints, it is more efficient to solve the dual since, in general, an additional constraint requires more computational effort than an additional variable. The solution procedure for a dual problem is on the similar lines as for a primal. However, in the case of primal simplex, feasibility is maintained throughout while the dual simplex starts with an infeasible solution while maintaining optimality.

The dual problem of cost minimization may be viewed as a product maximization problem. Where the product is maximized by varying the y variables, the imputed cost of each constraint. These implicit values are the shadow prices of constraints. They define the marginal value of the contribution of each constraint to objective function, say how much more output can be obtained by relaxing a constraint by one unit. If price of a resource is less than its shadow price, it is desirable to buy that resource and expand the production. The value of slack variable at optimum solution indicates cost, in terms of lowering the output, of using any activity which is not included in the optimum solution.

11.0 POST OPTIMALITY ANALYSIS

Many times it is necessary to study the variation in the optimal solution resulting from the variations in the various parameters such as the cost coefficients, technological constants, or due to addition/deletion of variables or constraints etc. The study of the change in optimal solution due to these changes is known as the post optimality analysis. The following type of parameter changes affect the optimal solution:

- Changes in the RHS constants of constraint equations.
- Changes in the objective function coefficients.
- Changes in the coefficients in the constraints.
- Addition of new variables.
- Addition of new constraints.

Due to these changes, the optimal solution may change in the following ways:

- The optimal solution remains unchanged.
- The basic variables remain the same, but their values change.
- The basic variables as well as their values change.

In most of the cases, it is not necessary to solve the problem from the beginning, and the final simplex tableau can be used to get the requisite answer.

12.0 IMPORTANT CLASS OF LP PROBLEMS

Many day-to-day linear programming engineering problems have some unique features which allow the use of special techniques for solution. Some of these are briefly discussed below:

12.1 Transportation Problem

The objective of a transportation problem is minimization of the cost of transporting a certain commodity from a number of origins to a number of destinations. This problem can also be formulated as a regular LP problem and solved using the simplex method. However, its special structure allows a more efficient and convenient procedure for its solution.

12.2 Network Flow Problems

A network is a configuration which consists of nodes joined by directed arcs. Each arc can support a flow. Associated with each arc are three parameters: a lower bound on the flow, an upper bound on the flow, and a cost value which represents the expense to be incurred in moving one unit of flow along the arc. A circulation is a set of flows in the network which preserves conservation of flow at each node. This means the total flow into a node must equal the total flow out of the node.

The objective of a network flow optimization algorithm is to generate a circulation in the network which minimizes the total system cost (defined as the sum of the flow in each arc times the arc expense) subject to the capacity restrictions on the flow in each arc. Thus, problems can be written as:

$$\text{Minimize } \Sigma\, C_i\, x_i \qquad\qquad\qquad ...(50)$$
$$\text{ST} \quad l_i \le x_i \le u_i \qquad\qquad\qquad ...(51)$$
$$x_{in} = x_{out} \qquad\qquad\qquad ...(52)$$

where, x_i = flow in arc i, C_i = cost value for arc i, l_i = lower bound on arc i, u_i = upper bound on arc i, x_{in} = sum of flows into node j, and x_{out} = sum of flows out of node j.

The out-of-kilter algorithm (OKA) is a general method for generating the optimum circulation in a capacitated cost network. To accomplish this, the OKA begins with any circulation, then maintaining a circulation in the network, it changes flows in an attempt to achieve the objective without violating any of the continuity and limit constraints. The OKA establishes a pricing system which assigns a price to each node.

12.3 Computer Codes for LP Problems

Now-a-days, very efficient computer packages are available for solution of LP problems which make the use of this technique very attractive. Other topics of interest in linear programming include the transportation problem and the assignment problem. For more on the linear programming and other optimization techniques, readers may refer Rao(1979) and Taha(1976).

13.0 REFERENCES

Rao, S.S., 1979. Optimization, Theory and Practice. Wiley Eastern Limited, New Delhi.

Taha, H.A., 1976. Operations Research: An Introduction. Macmillan Publishing Company, New York.

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