A Mathematical Groundwater Flow Model for Assessing Surface Water and Groundwater Interaction for a Partially Penetrating Stream

1.0 Introduction

A mathematical groundwater flow model has been described here to find the exchange of flow between a partially penetrating river and a homogeneous infinite aquifer. The model takes into account the changes in river stage ,and effect of pumping well. Given the values of aquifer parameters, the transmissivity and the storage coefficient, the saturated thickness below the river bed, saturated thickness far away from the river, the perimeter of the river reaches , the model can predict the exchange flow rate between the aquifer and the river reaches consequent to passage of a single or several successive floods and pumping by several wells.

2.0 Statement of the Problem

A schematic section of a partially penetrating river in a homogeneous and isotropic aquifer of infinite areal extent is shown in Fig.1. The aquifer is initially at rest condition. Due to passage of a flood, the river stages change with time. The changes are identical over a long reach of the river. Some wells also start withdrawing water from the aquifer. It is required to find the recharge from the river to the aquifer and the flow from the aquifer to the river after the recession of the flood.

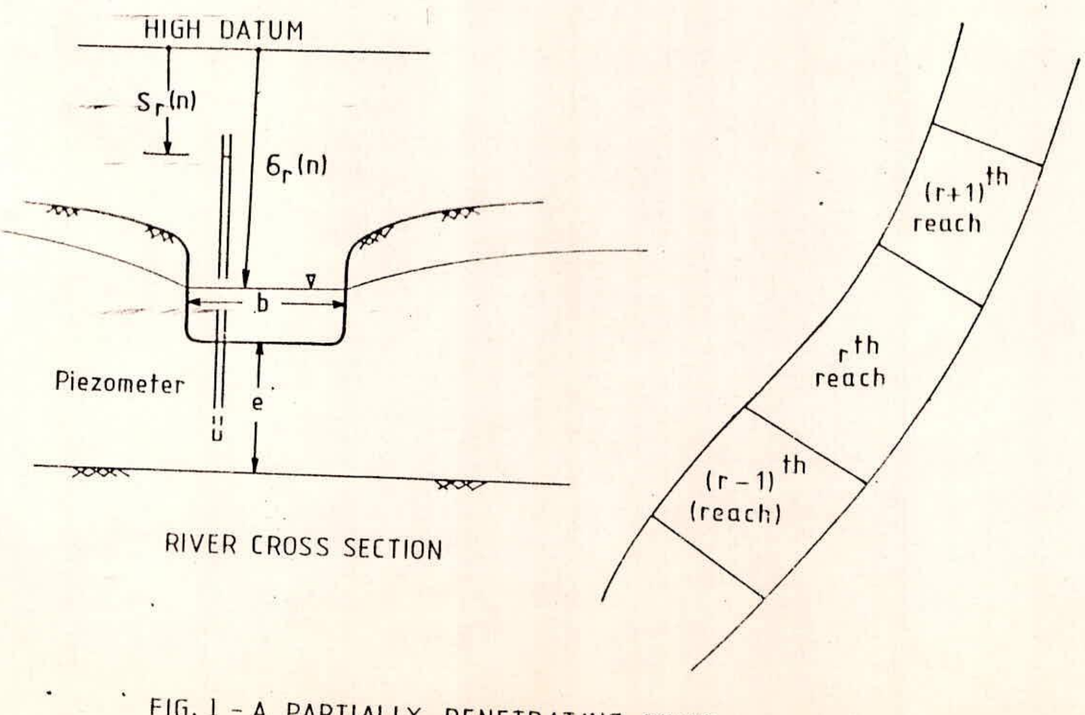

FIG. 1 - A PARTIALLY PENETRATING RIVER

Lecture delivered by Dr.G.C.Mishra, Scientist 'F', NIH, Roorkee.

3.0 Analysis

The following assumptions are made for the analysis:

i) The flow in the aquifer is in horizontal direction and two dimensional Boussinesq's equation governs the flow in the aquifer.

ii) The time parameter is discrete. Within each time step,the river stage, and the exchange flow rate between the river and the aquifer are separate constants but they very from step to step.

iii) The exchange of flow between the river and the aquifer is linearly proportional to the difference in the potentials at the river boundary and in the aquifer below the river bed.

iv) Since the governing differential equation is linear the method of superposition and proportionality are valid.

The differential equation which governs the flow in the aquifer is

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = \frac{\Phi}{T} \frac{\partial s}{\partial t} \qquad \ldots(1)$$

where

s = drawdown in the piezometric surface, T = transmissivity of the aquifer, and Φ = storage coefficient of the aquifer.

The aquifer being initially at rest condition, the initial condition to be satisfied is :

$s(x,0) = 0$

The boundary conditions to be satisfied are:

$s(\infty,t) = 0$, and at the river aquifer interface exchange of flow takes place which depends on the potential difference between the river and the aquifer

Let the time parameter be discretised by uniform time steps. Each time step may be 1 hour, one day or two weeks. It is assumed that the exchange flow rate within a time step is constant but it varies from step to step.Also it is assumed that pumping rate at a well is constant within a time step but varies from step to step.

The stream is divided into a number of reaches. The stream stage in a reach within a time step is also assumed to be constant but the stage in a reach varies with time step.

Let us consider the r^{th} reach. The flow from the aquifer to the r^{th} reach is given by

$$Q_r(n) = \Gamma_r(\sigma_r(n)-S_r(n)) \tag{2}$$

in which Γ_r is the reach transmissivity $\sigma_r(n)$ = stage of the stream reach measured from a high datum, $S_r(n)$ is the depth to piezometric surface measured from the same high datum. The reach transmissivity is given by

$$\Gamma_r = L_r T \frac{0.5w_p + e}{e(0.5e+4b)} \tag{3}$$

in which w_p is the wetted perimeter ,e is the thickness of the aquifer below the river bed ,b is the width of the river at the water surface, L_r is the reach length. $\sigma_r(n)$ is the stream stage which is known from observation of the stage.

At any time $S_r(n)$ is given by

$$S_r(n) = s_r(n) + S_o \tag{4}$$

in which $s_r(n)$ is the drawdown measured from the initially rest water table position. S_o is the depth to the piezometric surface below the stream when the stream and the aquifer were at rest condition.

$s_r(n)$ is given by

$$s_r(n) = \sum_{p=1}^{P} \sum_{\gamma=1}^{n} q_p(\gamma)\delta_{rp}(n-\gamma-1)$$

$$+ \sum_{\rho=1}^{R} \sum_{\gamma=1}^{n} Q_\rho(\gamma)\delta_{r\rho}(n-\gamma+1) \tag{5}$$

$q_p(\gamma)$ is the pumping rate at the p^{th} well during time step γ. There are P number of wells, $Q_\rho(\gamma)$ is the flow to the ρ^{th} reach during time step γ and there are R number of reaches which are hydraulically connected with the aquifer and are receiving water from the aquifer. The discrete pumping kernel is given by

$$\delta_{rp}(n) = \frac{1}{4\pi T}\left(E_1\left(\frac{r_{rp}^2}{4\beta n}\right) - E_1\left(\frac{r_{rp}^2}{4\beta(n-1)}\right)\right) \quad . \tag{6}$$

in which

T = transmissivity per unit time step ,

β = T/ϕ ,

ϕ = storage coefficient ,

r_{rp} = distance between the p^{th} pumping well and the r^{th} reach,

$E_1(x)$ = exponential integral

$$= \int_x^{\infty} \frac{e^{-u}}{u} du$$

$$E_1(x) + \ln x = a_o + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \varepsilon(x) \tag{7}$$

$$|\varepsilon(x)| < 2 \times 10^{-7}$$

a_o = −.57721 566	a_3 = .05519 968
a_1 = .99999 193	a_4 = .00976 004
a_2 = −.24991 055	a_5 = .00107 857

$$\delta_{rr}(n) = \frac{1}{\phi ab} \int_0^1 \operatorname{erf}\{a / (4 (\beta(n-\tau))^{1/2})\} \operatorname{erf}\{b / (4 (\beta(n-\tau))^{1/2})\} d\tau \tag{8}$$

$$\operatorname{erf}(Y) = \frac{2}{\sqrt{\pi}} \int_0^Y e^{-v^2} dv$$

$$\delta_{r\rho}(n) = \frac{1}{4\pi T}\left(E_1\left(\frac{r_{r\rho}^2}{4\beta n}\right) - E_1\left(\frac{r_{r\rho}^2}{4\beta(n-1)}\right)\right)$$

$r_{r\rho}$ = distance of the r^{th} reach from the ρ^{th} reach.

Equation (2) now can be written as:

$$Q_r(n) = \Gamma_r [\alpha_r(n) - (S_o + \sum_{p=1}^{P} \sum_{\gamma=1}^{n} q_p(\gamma)\delta_{rp}(n-\gamma+1)$$

$$+\sum_{\rho=1}^{R}\sum_{\gamma=1}^{n} Q_{\rho}(\gamma)\delta_{r\rho}(n-\gamma+1)\}] \qquad (9)$$

$Q_{\rho}(\gamma)$ are unknown for all ρ and all γ value.

Splitting the temporal summation into two parts, one part containing the summation up to $(n-1)^{th}$ time step and the other part containing the n^{th} term ,

$$\frac{1}{\Gamma_r} Q_r(n)+\sum_{\rho=1}^{R} Q_{\rho}(n)\delta_{r\rho}(1)=\sigma_r(n)-\{S_o+\sum_{p=1}^{P}\sum_{\gamma=1}^{n} q_p\delta_{rp}(n-\gamma+1)$$

$$+\sum_{\rho=1}^{R}\sum_{\gamma=1}^{n-1} Q_{\rho}(\gamma)\delta_{r\rho}(n-\gamma+1)\} \qquad (10)$$

Let the r^{th} reach, $(r+1)^{th}$ reach and $(r-1)^{th}$ reach be hydraulically connected,and no exchange of flow takes place through other reaches. Hence,

$$Q_r(n)\left[\frac{1}{\Gamma_r}+\delta_{rr}(1)\right]+Q_{r+1}(n)\,\delta_{r,r+1}(1)+Q_{r-1}(n)\delta_{r,r-1}(1)$$

$$+Q_{r-1}(n)\delta_{r,r-1}(1)$$

$$=\sigma_r(n)-\{S_o+\sum_{p=1}^{P}\sum_{\gamma=1}^{n} q_p(\gamma)\delta_{rp}(n-\gamma+1)$$

$$+\sum_{\gamma=1}^{n-1} Q_r(\gamma)\delta_{rr}(n-\gamma+1)$$

$$+\sum_{\gamma=1}^{n-1} Q_{r+1}(\gamma)\delta_{r,r+1}(n-\gamma+1)$$

$$+\sum_{\gamma=1}^{n-1} Q_{r-1}(\gamma)\delta_{r,r-1}(n-\gamma+1) \qquad (11)$$

Writing the equation for the $(r-1)^{th}$ reach

$$Q_{r-1}[\frac{1}{\Gamma_{r-1}} + \delta_{r-1,r-1}(1)] + Q_r(n)\delta_{r-1,r}(1) + Q_{r+1}(n)\delta_{r-1,r+1}(1)$$

$$= \Gamma_{r-1}(n) - \{S_0 + \sum_{p=1}^{P}\sum_{\gamma=1}^{n} q_p(\gamma)\delta_{r-1,p}(n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n-1} Q_r(\gamma)\delta_{r-1,r}(n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n-1} Q_{r-1}(\gamma)\delta_{r-1,r-1}(n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n-1} Q_{r+1}(\gamma)\delta_{r-1,r-1}(n-\gamma+1) \qquad (11)$$

Similarly the equation for the $(r+1)^{th}$ reach is

$$Q_{r+1}[\frac{1}{\Gamma_{r+1}} + \delta_{r+1,r+1}(1)] + Q_r(n)\delta_{r+1,r}(1) + Q_{r-1}(n)\delta_{r+1,r-1}(1)$$

$$= \alpha_{r+1}(n) - \{S_0 + \sum_{p=1}^{P}\sum_{\gamma=1}^{n} q_p(\gamma)\delta_{r+1,p}(n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n-1} Q_r(\gamma)\delta_{r+1,r}(n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n-1} Q_{r+1}(\gamma)\delta_{r+1,r+1}(n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n-1} Q_{r-1}(\gamma)\delta_{r+1,r-1}(n-\gamma+1)\} \qquad (13)$$

From equation 11,12,13 the three unknown $Q_{r-1}(n)$, $Q_r(n)$, $Q_{r+1}(n)$, can be solved simultaneously in succession starting from the first time step. For n=1 the terms $\sum_{\gamma=1}^{n-1} Q_r(\gamma)\delta_{r\rho}(n-\gamma+1)=0$.

Thus the flow contributed to each reach can be estimated.

REFERENCES

Bouwer,H. 1969. Theory of seepage from open channels.Advances in Hydroscience, edited by V.T.Chow, Academic Press, V.5,pp.121-172.

Cooper, H.H.,Jr.,and M.I. Rorabough,1963. Ground Water Movements and Bank Storage Due to Flood Stages in Surface Streams,Geol.,Surv.Water Supply Paper 1536-J.

Morel-Seytoux,H.J. 1975. Optimal Legal Conjunctive Operation of Surface and Ground Waters. Proceedings Second World Congress,International Water Resources Association, New Delhi.

Morel-Seytoux,H.J. and C.J.Daly,1977. A Discrete Kernel Generator for Stream Aquifer Studies. Water Resources Research Journal.

Todd,D.K.,1955. Ground Water Flow in Relation to a Flooding Stream ,Proc.Amer.Soc .Civil Engineers, Vol.81 Sep.628 pp.20.