NATIONAL INSTITUTE OF HYDROLOGY
ROORKEE

WORKSHOP

ON

GROUND WATER MODELLING-TYSON-WEBER MODEL
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LECTURE	:	IV
TOPIC	:	STABILITY AND CONVERGENCE
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CONVERGENCE AND STABILITY

By

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The numerical solutions of the partial differential equations governing transient groundwater flow, generally involve approximation of spatial and temporal derivatives by Taylor's series expansion. This approximation will always lead to certain 'truncation errors' since only the first few terms of the otherwise infinite series are considered and the terms containing higher derivatives are neglected. The magnitude of errors will depend upon the behaviour of higher derivatives as well as the step sizes in space and time. If the higher derivatives of the true function are 'well behaved' (i.e. close to zero) one would normally expect the errors to attenuate with lowering of step sizes provided the round-off errors can be contained. However unfortunately, the attenuation of errors is derivatives need not always imply better estimates of the solution i.e. hydraulic heads. Similarly the errors incurred at each time step may accumulate and the resulting errors at large times may aggravate severely. With these concepts we are ready to study semi-formal definitions of convergence

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and stability.

Let us consider the computation of piezomatric head distribution at time t_2, commencing from a known distribution at time t_1. Let us further assume that the known distribution at time t_1 is completely error free. If ($\Delta x, \Delta y$) and Δt are the adopted space and time steps respectively then n, the number of time steps required to reach t_2 time level will be $(t_2-t_1)/\Delta t$. If (H) is the true solution (unknown) and (h) is the computed solution at time t_2, then the adopted scheme can be considered to be 'convergent' provided.

$$\lim_{\substack{x \to 0 \\ y \to 0 \\ t \to 0 \\ n \to \infty}} |(H) - (h)| = 0$$

It is evident that the property of convergence is quite crucial. One can expect to reach better solutions (closer to the true solution) by reducing space and time steps only if the adopted scheme is convergent. On the other hand the reduction of the space and time steps (associated with increased computation efforts) can be counter productive if convergence is not ensured.

It can be inferred from the preceding discussions that there will always be some residual errors on account of finite step sizes even if the scheme is convergent. Thus, one is concerned with the possible accumulation of these errors in time domain. In case the errors tend to accumulate montonously (without any compensation) the scheme is said to be 'unstable'. Thus, the stability can be defined in terms of the error propagation in time domain, as follows:

$$\underset{t \to \infty}{Lt} \quad | \; (H) \; - \; (h) \; | \quad \leq M$$

where M is a finite non zero positive number. This condition stipulates that the error is bounded by M(how so large it may be, depending upon the truncation error incured at each time step). It is essential to ensure stability especially when dealing with long term predictions.

Unlike explicit schemes, the implicit schemes are unconditionally stable and convergent. However, in certain simple cases the explicit schemes can be designed for stability and convergence. This would indeed be quite desirable since the computational efforts involved in explicit solutions are significantly lower. However, the criterion for their stability and convergence is known exactly only for the case of one dimensional flow in homogenous aquifer discretised by uniformly spaced nodal points . The criterion is

$$\frac{T}{S} \times \frac{\Delta t}{(\Delta x)^2} < \frac{1}{2}$$

For two and three dimensional finite difference models with varying nodal spacings there is no way of ensuring stability and convergence for explicit schemes. Hence it is a normal practice to employ implicit schemes inspite of higher computation efforts.