Determination of Aquifer Parameters

G.C. MISHRA
Water Resources Development and Management, IIT Roorkee

INTRODUCTION

Predicting response of an aquifer to a pumping pattern, whose transmissivity and storage coefficient are known a priori, is classified as a direct problem. Estimating the aquifer parameters from a set of observed response of the aquifer to a known pumping pattern is an inverse problem. An inverse problem could be solved provided the corresponding direct problem has been solved a priori.

Using Theis' basic solution i.e. by matching the time drawdown curve with Theis' type curves, we can determine the parameters of an aquifer which is confined, homogeneous, isotropic, and is of infinite area and the pumping well has small radius. The aquifer is to be initially at rest condition, and the aquifer test is conducted under constant pumping rate. These are the assumptions on which Theis' solution is based.

Mishra Chachadi's type curves (Mishra, Chachadi, 1986) could be used for determining parameters of a confined aquifer if the test is conducted in a large diameter well. These type curves include both the pumping phase and recovery the recovery phase

To avoid human error while curve matching, the inverse problem could be solved conveniently using Marquardt Algorithm (Marquardt, 1963). Berg (1971), and Chander et.al.(1981) have used the algorithm to predict parameters of aquifers in different hydro geological settings.

In a hard rock region in Jodhpur city, aquifer tests have been conducted in small radius tube wells and in large diameter wells locally known as Bowaris. In this study, for the aquifer test conducted in the wells with small radie, the inverse problem has been solved using the Theis' basic solution treating the aquifer to be confined. For the tests conducted in large diameter wells, Hantush' basic solution for well with finite radius has been used considering well storage effect on drawdown data. Unit pulse kernel coefficients are generated and used in a Marquardt Algorithm as described below. The convolution technique described here is quite versatile. The aquifer test may be conducted under constant pumping rate, or under variable pumping rate as in step draw down test. One could also determine the aquifer parameters using the recovery data.

THE MARQUARDT ALGORITHM

The Theis' solution, which provides evolution of drawdown in a confined aquifer in response to constant continuous pumping from a fully penetrating well with small radius having negligible well storage, is

$$s(Q,T,\phi,r,t)=\left\{\frac{Q}{4\pi T}\right\}\left[\int_{r^2\phi/(4Tt)}^{\infty}\frac{e^{-u}}{u}du\right]=F_1\left(Q,r,t,T,\phi\right) \quad (1)$$

where $s\left(Q,T,\phi,r,t\right)=$ is the draw down in piezometric surface at distance r and time t after the onset of pumping ; $Q=$ constant pumping rate; $T=$ transmissivity of the confined aquifer, and $\phi=$storage coefficient.

Partial derivative of $s(Q,T,\phi,r,t)$ with respect to T is

$$\frac{\partial s(Q,T,\phi,r,t)}{\partial T}=\left\{\frac{-Q}{4\pi T^2}\right\}\left[\int_{r^2\phi/(4Tt)}^{\infty}\frac{e^{-u}}{u}du\right]+\left\{\frac{Q}{4\pi T}\right\}\left[-\left(\frac{e^{-r^2\phi/(4Tt)}}{r^2\phi/(4Tt)}\right)\left(-\frac{r^2\phi}{4T^2t}\right)\right]$$
$$=\left\{\frac{Q}{4\pi T^2}\right\}\left[e^{-r^2\phi/(4Tt)}-\int_{r^2\phi/(4Tt)}^{\infty}\frac{e^{-u}}{u}du\right]=F_2(Q,r,t,T,\phi)\quad(2)$$

Partial derivative of $s(Q,T,\phi,r,t)$ with respect to ϕ is

$$\frac{\partial s(Q,T,\phi,r,t)}{\partial \phi}=\left\{\frac{Q}{4\pi T}\right\}\left[-\left(\frac{e^{-r^2\phi/(4Tt)}}{r^2\phi/(4Tt)}\right)\left(\frac{r^2}{4Tt}\right)\right]=-\frac{Q}{4\pi T\phi}e^{-r^2\phi/(4Tt)}=F_3(Q,r,t,T,\phi)\quad(3)$$

$F_1(Q,r,t,T,\phi),F_2(Q,r,t,T,\phi),F_3(Q,r,t,T,\phi)$ are functions of T ϕ Q, r, and t.

Let T^* and ϕ^* be the approximate values of transmissivity and storage coefficient near to the true values of transmissivity and storage coefficient. Initially, T^* and ϕ^* are to be guessed. Let ΔT and $\Delta\phi$ be incremental values in transmissivity and storage coefficient so that $T^*+\Delta T$ and $\phi^*+\Delta\phi$ are nearer to the true values. ΔT and $\Delta\phi$ are unknown and are to be predicted by Marquardt algorithm. Performing Taylor series expansion of drawdown $s(Q,T,\phi,r,t)$ about T^* and ϕ^*, and neglecting the higher order terms

$$s(Q,T,\phi,r,t)\Big|_{T^*+\Delta T,\phi^*+\Delta\phi}=s(Q,T,\phi,r,t)\Big|_{T^*,\phi^*}+\frac{\partial s(Q,T,\phi,r,t)}{\partial T}\Big|_{T^*,\phi^*}\Delta T+\frac{\partial s(Q,T,\phi,r,t)}{\partial \phi}\Big|_{T^*\phi^*}\Delta\phi$$
$$=F_1(Q,r,t,T^*,\phi^*)+F_2(Q,r,t,T^*,\phi^*)\Delta T+F_3(Q,r,t,T^*,\phi^*)\Delta\phi\quad(4)$$

The pumping rate Q and the distance r of the piezometer from the pumping well being constants, we abbreviate $s(Q,T,\phi,r,t)$ by $s(t,T,\phi)$, $F_1(Q,r,t,T^*,\phi^*)$ by $F_1(t,T^*,\phi^*)$, $F_2(Q,r,t,T^*,\phi^*)$ by $F_2(t,T^*,\phi^*)$ and $F_3(Q,r,t,T^*,\phi^*)$ by $F_3(t,T^*,\phi^*)$.

Let $s_o(i)$ be the i^{th} observed drawdown in piezometric surface in the piezometer at time t_i. The predicted drawdown , $s_c(t_i,T^*+\Delta T,\phi^*+\Delta\phi)$, at observation time t_i from equation (4) is

$$s_c(t_i,T^*+\Delta T,\phi^*+\Delta\phi)=F_1(t_i,T^*,\phi^*)+F_2(t_i,T^*,\phi^*)\Delta T+F_3(t_i,T^*,\phi^*)\Delta\phi\quad(5)$$

The error in the i^{th} prediction, $E(i)$, is

$$E(i)=s_o(i)-s_c(t_i,T^*+\Delta T,\phi^*+\Delta\phi)$$
$$=s_o(i)-\left\{F_1(t_i,T^*,\phi^*)+F_2(t_i,T^*,\phi^*)\Delta T+F_3(t_i,T^*,\phi^*)\Delta\phi\right\}\quad(6)$$

The Marquardt algorithm minimizes sum of the squares of error for a set of N observations and the minimization problem is

$$\underset{\Delta T,\,\Delta\phi}{Min}\left\{\sum_{i=1}^{N}\left[s_0(i)-\left\{F_1\left(t_i,T^*,\phi^*\right)+F_2\left(t_i,T^*,\phi^*\right)\Delta T+F_3\left(t_i,T^*,\phi^*\right)\Delta\phi\right\}\right]^2\right\} \tag{7}$$

Differentiating sum of the squares of the error with respect to ΔT and equating it to zero

$$\sum_{i=1}^{N}\left\{-2\left[s_0(i)-\left\{F_1\left(t_i,T^*,\phi^*\right)+F_2\left(t_i,T^*,\phi^*\right)\Delta T+F_3\left(t_i,T^*,\phi^*\right)\Delta\phi\right\}\right]F_2\left(t_i,T^*,\phi^*\right)\right\}=0 \tag{8}$$

Simplifying, equation (8) reduces to

$$\left\{\sum_{i=1}^{N}\left[F_2\left(t_i,T^*,\phi^*\right)F_2\left(t_i,T^*,\phi^*\right)\right]\right\}\Delta T+\left\{\sum_{i=1}^{N}\left[F_3\left(t_i,T^*,\phi^*\right)F_2\left(t_i,T^*,\phi^*\right)\right]\right\}\Delta\phi$$
$$=\sum_{i=1}^{N}\left[s_0(i)F_2\left(t_i,T^*,\phi^*\right)\right]-\sum_{i=1}^{N}\left[F_1\left(t_i,T^*,\phi^*\right)F_2\left(t_i,T^*,\phi^*\right)\right] \tag{9}$$

Differentiating sum of the squares of the error with respect to $\Delta\phi$ and equating it to zero

$$\sum_{i=1}^{N}\left\{-2\left[s_0(i)-\left\{F_1\left(t_i,T^*,\phi^*\right)+F_2\left(t_i,T^*,\phi^*\right)\Delta T+F_3\left(t_i,T^*,\phi^*\right)\Delta\phi\right\}\right]F_3\left(t_i,T^*,\phi^*\right)\right\}=0 \tag{10}$$

Simplifying, equation (10) reduces to

$$\left\{\sum_{i=1}^{N}\left[F_2\left(t_i,T^*,\phi^*\right)F_3\left(t_i,T^*,\phi^*\right)\right]\right\}\Delta T+\left\{\sum_{i=1}^{N}\left[F_3\left(t_i,T^*,\phi^*\right)F_3\left(t_i,T^*,\phi^*\right)\right]\right\}\Delta\phi$$
$$=\sum_{i=1}^{N}\left[s_0(i)F_3\left(t_i,T^*,\phi^*\right)\right]-\sum_{i=1}^{N}\left[F_1\left(t_i,T^*,\phi^*\right)F_3\left(t_i,T^*,\phi^*\right)\right] \tag{11}$$

Equations (9) and (11) are written as

$$a(1,1)\Delta T+a(1,2)\Delta\phi=c(1) \tag{12}$$

$$a(2,1)\Delta T+a(2,2)\Delta\phi=c(2) \tag{13}$$

Solving for $\Delta\phi$ and ΔT, we get

$$\Delta\phi=\frac{\dfrac{c(1)}{a(1,1)}-\dfrac{c(2)}{a(2,1)}}{\dfrac{a(1,2)}{a(1,1)}-\dfrac{a(2,2)}{a(2,1)}} \tag{14}$$

and

$$\Delta T = \frac{c(1)}{a(1,1)} - \frac{a(1,2)}{a(1,1)} \Delta\phi \tag{15}$$

where

$$a(1,1) = \sum_{i=1}^{N} \left[F_2\left(t_i, T^*, \phi^*\right) F_2\left(t_i, T^*, \phi^*\right) \right]; \quad a(1,2) = \sum_{i=1}^{N} \left[F_3\left(t_i, T^*, \phi^*\right) F_2\left(t_i, T^*, \phi^*\right) \right]$$

$$c(1) = \sum_{i=1}^{N} \left[s_0(i) F_2\left(t_i, T^*, \phi^*\right) \right] - \sum_{i=1}^{N} \left[F_1\left(t_i, T^*, \phi^*\right) F_2\left(t_i, T^*, \phi^*\right) \right]$$

and

$$a(2,1) = \sum_{i=1}^{N} \left[F_2\left(t_i, T^*, \phi^*\right) F_3\left(t_i, T^*, \phi^*\right) \right]; \quad a(2,2) = \sum_{i=1}^{N} \left[F_3\left(t_i, T^*, \phi^*\right) F_3\left(t_i, T^*, \phi^*\right) \right]$$

$$c(2) = \sum_{i=1}^{N} \left[s_0(i) F_3\left(t_i, T^*, \phi^*\right) \right] - \sum_{i=1}^{N} \left[F_1\left(t_i, T^*, \phi^*\right) F_3\left(t_i, T^*, \phi^*\right) \right]$$

The improved transmissivity and storage coefficient are given by

$$T^*\big|_{new} = T^*\big|_{old} + \Delta T \tag{16}$$

$$\phi^*\big|_{new} = \phi^*\big|_{old} + \Delta\phi \tag{17}$$

This iteration procedure is to be repeated till ΔT and $\Delta\phi$ tend to very small values.

Evaluation of the Functions $F_1\left(t_i, T^*, \phi^*\right)$ and $F_2\left(t_i, T^*, \phi^*\right)$

The functions $F_1\left(Q, r, t_i, T^*, \phi^*\right)$ and $F_2\left(Q, r, t_i, T^*, \phi^*\right)$ are evaluated for different observation time t_i using a convolution technique. Let the observation time span be discretized by uniform time steps. In case of aquifer test, a convenient time step size is one minute as observations are generally made at different intervals of minutes. Accordingly, in the functions $F_1\left(Q, r, t_i, T^*, \phi^*\right)$, $F_2\left(Q, r, t_i, T^*, \phi^*\right)$ and $F_3\left(Q, r, t_i, T^*, \phi^*\right)$, the unit of pumping rate is m^3 per minute, and unit of transmissivity is m^2 per minute. Let $Q(i)$ be pumping rate during i^{th} time period. During recovery period, the pumping rate is zero. Under variable pumping rate, $Q(t)$, the drawdown $s(Q(i), T^*, \phi^*, r, n)$ at the end of n minutes is derived as follows (Morel- Seytoux, 1975)

$$s(Q(t), T^*, \phi^*, r, n) = \int_0^n \frac{Q(\tau)}{4\pi T^*(n-\tau)} e^{-\frac{r^2\phi^*}{4T^*(n-\tau)}} d\tau$$

$$= \int_0^1 \frac{Q(1)}{4\pi T^*(n-\tau)} e^{-\frac{r^2\phi^*}{4T^*(n-\tau)}} d\tau + \int_1^2 \frac{Q(2)}{4\pi T^*(n-\tau)} e^{-\frac{r^2\phi^*}{4T^*(n-\tau)}} d\tau + \ldots$$

$$+\int_{\gamma-1}^{\gamma}\frac{Q(\gamma)}{4\pi T^{*}(n-\tau)}e^{-\frac{r^{2}\phi^{*}}{4T^{*}(n-\tau)}}d\tau..,+\int_{n-1}^{n}\frac{Q(n)}{4\pi T^{*}(n-\tau)}e^{-\frac{r^{2}\phi^{*}}{4T^{*}(n-\tau)}}d\tau$$

$$=\sum_{\gamma=1}^{n}Q(\gamma)\delta(n-\gamma+1) \qquad (18)$$

The unit response function coefficient $\delta(m)$ is given by:

$$\delta(m)=\int_{0}^{m}\frac{1}{4\pi T^{*}(m-\tau)}e^{-\frac{r^{2}\phi^{*}}{4T^{*}(m-\tau)}}d\tau-\int_{0}^{m-1}\frac{1}{4\pi T^{*}(m-1-\tau)}e^{-\frac{r^{2}\phi^{*}}{4T^{*}(m-1-\tau)}}d\tau$$

$$=\frac{1}{4\pi T^{*}}\int_{\frac{r^{2}\phi^{*}}{4T^{*}m}}^{\infty}\frac{e^{-u}}{u}du-\frac{1}{4\pi T^{*}}\int_{\frac{r^{2}\phi^{*}}{4T^{*}(m-1)}}^{\infty}\frac{e^{-u}}{u}du$$

$$=\frac{1}{4\pi T^{*}}W\left(\frac{r^{2}\phi^{*}}{4T^{*}m}\right)-\frac{1}{4\pi T^{*}}W\left(\frac{r^{2}\phi^{*}}{4T^{*}(m-1)}\right) \qquad (19)$$

$W\left(\frac{r^{2}\phi^{*}}{4T^{*}m}\right)$ is Theis' Well function with argument $\frac{r^{2}\phi^{*}}{4T^{*}m}$. $W\left(\frac{r^{2}\phi^{*}}{4T^{*}m}\right)$ is an exponential integral. For argument *X*, the exponential integral, $W(X)$, is computed using the following polynomial approximation.

For $X\le 1$

$$W(X)=-\ln(\mathrm{X})-0.57721566+0.99999193\mathrm{X}-0.24991055\mathrm{X}^{2}$$
$$+0.05519968*\mathrm{X}^{3}-0.00976004\mathrm{X}^{4}+0.00107857\mathrm{X}^{5} \qquad (20)$$

For $X>1$

$$Xe^{X}W(X)=\frac{\mathrm{X}^{4}+8.5733287\mathrm{X}^{3}+18.059017\mathrm{X}^{2}+8.6347608\mathrm{X}+0.26777373}{\mathrm{X}^{4}+9.5733223\mathrm{X}^{3}+25.632956\mathrm{X}^{2}+21.099653\mathrm{X}+3.9584969} \qquad (21)$$

An Example

A set of synthetic observation data generated using $T=0.1m^{2}/\min$ and storage coefficient $\phi=0.001$ are as given in Table 1. Predict the T,ϕ making an initial guess $T^{*}=0.01m^{2}/\min$ and $\phi^{*}=0.003$. The piezometer is located at a distance of $20m$ from the pumping well. The pumping rate is $Q=0.2m^{3}/\min$.

Table1: Synthetic Drawdown Data Generated Using Equation (18)

Time of observation (min)	Observed Drawdown(m)	Time of observation (min)	Observed Drawdown(m)
1	0.035	60	0.562
2	0.089	70	0.587
3	0.132	80	0.608
4	0.166	90	0.626
5	0.195	100	0.643
6	0.219	120	0.671
7	0.24	140	0.696
8	0.258	160	0.717
9	0.275	180	0.736
10	0.29	200	0.752
12	0.317	230	0.774
14	0.339	260	0.794
16	0.359	290	0.811
18	0.377	320	0.827
20	0.393	350	0.841
25	0.427	380	0.854
30	0.455	410	0.866
35	0.478	440	0.877
40	0.499	470	0.888
50	0.534	480	0.891

Table 2: Convergence of T^* and ϕ^* with Successive Iteration

Iteration no	T^*	ϕ^*	ΔT	$\Delta\phi$	C(1)	C(2)
1	0.020507	0.004138666	1.05E-02	1.14E-03	0.00E+00	3.64E-12
2	0.040832	0.003697811	2.03E-02	-4.41E-04	-1.14E-13	0.00E+00
3	0.07088	0.001491592	3.00E-02	-2.21E-03	1.42E-14	0.00E+00
4	0.092492	0.001075698	2.16E-02	-4.16E-04	0.00E+00	-1.14E-13
5	0.099488	0.001004064	7.00E-03	-7.16E-05	0.00E+00	1.42E-14
6	0.099998	0.001000018	5.10E-04	-4.05E-06	0.00E+00	0.00E+00
7	0.1	0.001	2.42E-06	-1.81E-08	0.00E+00	0.00E+00
8	0.1	0.001	5.43E-11	-4.07E-13	0.00E+00	2.12E-22

Thus using synthetic drawdown data, we have checked that Marquardt Algorithm successfully predicts the true transmissivity and storage coefficient when the initial guess was different from the true value.

Determination of Aquifer Parameters Applying Marquardt Algorithm to Observed Drawdown at a Piezometer During Pumping

Location: Jodhpur, Paota. Pumping Rate, Q = 0.21 m^3 / min.

Distance of Observation Well from Pumping Well = 24.5m

Table 3 Observed drawdown

Time of observation (min)	Observed Drawdown(m)	Time of observation (min)	Observed Drawdown(m)
1	0	60	1.65
2	0.08	70	1.719
3	0.13	80	1.762
4	0.23	90	1.802
5	0.33	100	1.836
6	0.41	120	1.902
7	0.48	140	1.938
8	0.55	160	1.974
9	0.64	180	2.004
10	0.71	200	2.03
12	0.82	230	2.052
14	0.93	260	2.082
16	1.02	290	2.102
18	1.08	320	2.122
20	1.14	350	2.132
25	1.26	380	2.142
30	1.35	410	2.17
35	1.42	440	2.172
40	1.46	470	2.172
45	1.5	480	2.172
50	1.58		

Starting transmissivity value, T^* (m^2 / min) = 0 .010; starting storage coefficient, ϕ^* = 0.0003.

Table 4 Transmissivity and Storage Coefficient as Obtained through Successive Iteration

Iteration no	T^*	ϕ^*	ΔT	$\Delta\phi$	C(1)	C(2)
1	0.017997	0.000373	8.00E-03	7.33E-05	0.00E+00	0.00E+00
2	0.028653	0.000338	1.07E-02	-3.55E-05	0.00E+00	0.00E+00
3	0.036955	0.000276	8.30E-03	-6.21E-05	0.00E+00	0.00E+00
4	0.039092	0.00027	2.14E-03	-5.46E-06	0.00E+00	0.00E+00
5	0.039129	0.000272	3.65E-05	1.32E-06	0.00E+00	0.00E+00
6	0.039133	0.000271	4.43E-06	-9.17E-08	2.78E-17	8.88E-16
7	0.039132	0.000271	-5.05E-07	1.11E-08	0.00E+00	0.00E+00
8	0.039133	0.000271	6.01E-08	-1.31E-09	-4.34E-19	0.00E+00
9	0.039133	0.000271	-7.14E-09	1.56E-10	5.42E-20	0.00E+00

Iterated Transmissivity, T = 0 .03913 m^2 / min = 56.35 m^2 / *day*

Iterated Storage Coefficient, ϕ = 0.000271

Table 5 Comparison of Observed Drawdowns with Simulated Drawdowns

Time(min)	Observed drawdown(m)	Computed drawdown(m)	Time(min)	Observed drawdown(m)	Computed drawdown(m)
1	0	0.0875	60	1.65	1.4921
2	0.08	0.2287	70	1.719	1.5569
3	0.13	0.3417	80	1.762	1.6132
4	0.23	0.4326	90	1.802	1.6628
5	0.33	0.5081	100	1.836	1.7073
6	0.41	0.5725	120	1.902	1.7845
7	0.48	0.6285	140	1.938	1.8498
8	0.55	0.6781	160	1.974	1.9064
9	0.64	0.7226	180	2.004	1.9564
10	0.71	0.7629	200	2.03	2.0011
12	0.82	0.8337	230	2.052	2.0605
14	0.93	0.8945	260	2.082	2.1127
16	1.02	0.9477	290	2.102	2.1591
18	1.08	0.995	320	2.122	2.201
20	1.14	1.0376	350	2.132	2.2392
25	1.26	1.1285	380	2.142	2.2742
30	1.35	1.2035	410	2.17	2.3066
35	1.42	1.2672	440	2.172	2.3366
40	1.46	1.3227	470	2.172	2.3647
45	1.5	1.3717	480	2.172	2.3737
50	1.58	1.4158			

Square root of sum of square of error = 0. 82m
Average error = 0. 02 m

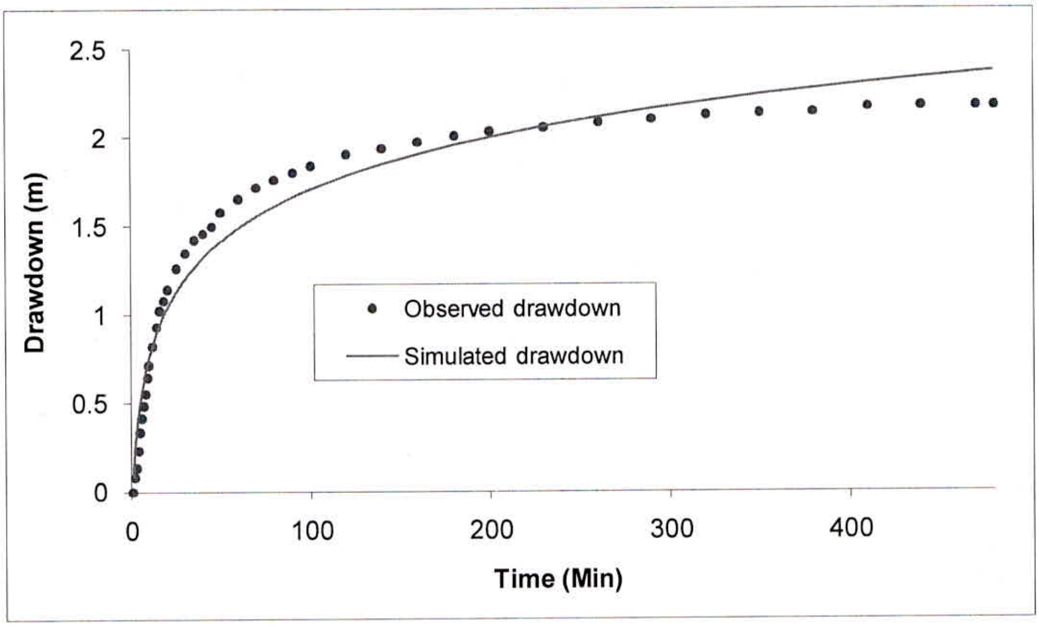

Fig. 1 Observed and Simulated drawdowns for T = 0 .03913 $m^2/\min$ =56.35 m^2/day and Storage Coefficient, ϕ = 0.000271

Determination of Transmissivity and Storage Coefficient Using Data of an Aquifer Test Conducted in a Large Diameter Well

In hard rock region, the shallow aquifers have low transmissivity ranging from 25 to 100 m^2 / day Therefore, in hard rock region, wells with diameter ranging from 1 to 2 m are constructed to have reasonable yield. An aquifer test can be conducted in a large diameter well and the recovery data can be used for a reasonable estimate of storage coefficient and transmissivity.

Solution to unsteady flow to a dug-cum-bore well, that takes well storage into account, has been derived by Papadopulos and Cooper (1967). According to them, the well storage dominates the time-drawdown curve up to a time, t, given by $t = \left(25 r_c^2\right) / T$ where r_c is radius of the well casing, and T is transmissivity of the aquifer. If a short duration aquifer test is conducted in a large diameter well, the transmissivity can be estimated reasonably well, but the storage coefficient may differ by an order of magnitude. This is because, the type curves presented by Papadopulos and Cooper contain straight line portions, which are parallel, and a short duration time-drawdown curve if matched with one of the straight lines, it could be matched as well with either of the adjacent straight lines. Discretising the time domain by uniform time steps, and generating unit response function coefficients from Thies' basic solution for unsteady flow to a well with small radius, Patel and Mishra (1983), Mishra and Chachadi (1985) have derived simple analytical solutions to unsteady flow during pumping, and during recovery respectively. These solutions, as well as that by Papadopulos and Cooper are applicable for a bore well with small radius having large casing.

Hantush has derived an analytical solution to unsteady flow to a well with finite radius assuming that all the water pumped is from aquifer storage. The effect of well storage on time drawdown curve has not been taken into account in the solution. Discretising the time domain by uniform time steps, and generating unit response function coefficients from Hantush's basic solution, we derive a simple analytical solutions to unsteady flow to a large diameter well during pumping and recovery. The well storage contribution during pumping, and well storage effect on drawdown have been accounted. After solving the direct problem, the inverse problem has been solved using the Marquardt Algorithm as described below. Pumping as well as recovery data could be used for estimating the aquifer parameters.

Let the total time of observation including pumping and recovery periods be discretised to N units of equal time steps of size Δt. Let the pumping continued until the end of m^{th} time step. During any time step n, the quantity of water pumped is sum of the quantity drawn from aquifer storage and the quantity drawn from well storage. Therefore,

$$\frac{Q_a\{(n-1)\Delta t\} + Q_a(n\Delta t)}{2}\Delta t + \frac{Q_w\{(n-1)\Delta t\} + Q_w(n\Delta t)}{2}\Delta t = \frac{Q_p\{(n-1)\Delta t\} + Q_p(n\Delta t)}{2}\Delta t \quad (22)$$

in which, $Q_a(n\Delta t)$, $Q_w(n\Delta t)$ and $Q_p(n\Delta t)$ are withdrawal rates from aquifer storage and well storage, and pumping rate respectively at time $t = n\Delta t$. The time step size Δt can be chosen conveniently incorporating $\frac{Q_a\{(n-1)\Delta t\} + Q_a(n\Delta t)}{2} = \bar{Q}_a(n)$

$\frac{Q_w\{(n-1)\Delta t\} + Q_w(n\Delta t)}{2} = \bar{Q}_w(n)$ and $\frac{Q_p\{(n-1)\Delta t\} + Q_p(n\Delta t)}{2} = \bar{Q}_p(n)$ in Eq. (22),

$$\bar{Q}_a(n)+\bar{Q}_w(n)=\bar{Q}_p(n) \tag{23}$$

Let the well discharge be constant equal to Q_p. For $n \le m, \bar{Q}_p(n) = Q_p$;and for $n > m,\ \ \bar{Q}_p(n) = 0$.

Drawdown, $s_w(n)$, at the well face at the end of time step n is given by (Patel and Mishra, 1983)

$$s_w(n\Delta t)=\frac{1}{\pi r_c^2}\sum_{\gamma=1}^{n}\bar{Q}_w(\gamma)\Delta t \tag{24}$$

in which, $\bar{Q}_w(\gamma)$ is average withdrawal rate from well storage during time step γ. $\bar{Q}_w(\gamma)$ values are unknown a priori. A negative value of $\bar{Q}_w(\gamma)$ means replenishment of well storage from aquifer storage during time of recovery. r_c is radius of well casing. For some well r_c is equal to well bore radius r_w.

Following Duhamels' principle and method of convolution, drawdown at the well face at the end of time step $n\ (t = n\Delta t)$ due to abstraction from aquifer storage is given by (Morel-Seytoux, 1975)

$$s_a(r_w,n\Delta t)=\sum_{\gamma=1}^{n}\bar{Q}_a(\gamma)\delta(r_w,\Delta t,n-\gamma+1) \tag{25}$$

where $\delta(r_w,\Delta t,N)$ is a unit pulse response function coefficient derived from unit step response function using a time step size Δt (Morel-Seytoux, 1975) , and N is an integer. The kernel coefficient $\delta(r_w,\Delta t,N)$ is given by

$$\delta(r_w,\Delta t,N)=\frac{1}{\Delta t}\left\{U(r_w,N\Delta t)-U(r_w,(N-1)\Delta t)\right\};\ N>1 \tag{26}$$

For $N=1,\ \delta(r_w,\Delta t,1)=\frac{1}{\Delta t}U(r_w,\Delta t)$.

For well with small radius $U(r,I\Delta t)$ is the drawdown corresponding to unit pumping rate which could be computed using Theis' solution. The time step size Δt can be chosen conveniently. In an aquifer test, draw down observations are made at different intervals of minutes. Therefore, for solving an inverse problem, it is convenient to choose $\Delta t = 1$ minute. Accordingly, pumping rate is to be chosen in m^3 per minute and transmissivity in m^2 per minute. For a well with finite radius, the unit step response function has been derived by Hantush as given in Appendix I.

Assuming that there is no surface of seepage at the well face, the drawdown in the well is equated to the drawdown in the aquifer at the well face i.e. $s_w(n) = S_a(r_w,n)$. Equating equations (24) and (25)

$$\frac{\Delta t}{\pi r_c^2}\sum_{\gamma=1}^{n}\bar{Q}_w(\gamma)=\sum_{\gamma=1}^{n}\bar{Q}_a(\gamma)\delta(r_w,\Delta t,n-\gamma+1) \tag{27}$$

From equation (23)

$$\bar{Q}_w(n)=\bar{Q}_p(n)-\bar{Q}_a(n) \tag{28}$$

Incorporating (7) in (6)

$$\frac{\Delta t}{\pi r_c^2}\sum_{\gamma=1}^{n}\{\bar{Q}_p(\gamma)-\bar{Q}_a(\gamma)\}=\sum_{\gamma=1}^{n}\bar{Q}_a(\gamma)\delta(r_w,\Delta t,n-\gamma+1) \tag{29}$$

Splitting each temporal summation into two parts, the first part up to $(n-1)^{th}$ step, and the second part the n^{th} step, incorporating $\Delta t=1$ minute and solving for $\bar{Q}_a(n)$

$$\bar{Q}_a(n)=\frac{\sum_{\gamma=1}^{n}\bar{Q}_p(\gamma)-\sum_{\gamma=1}^{n-1}\bar{Q}_a(\gamma)-\pi r_c^2\sum_{\gamma=1}^{n-1}\bar{Q}_a(\gamma)\delta(r_w,\Delta t,n-\gamma+1)}{1+\pi r_c^2\delta(r_w,\Delta t,1)} \tag{30}$$

In particular for time step $n=1$

$$\bar{Q}_a(1)=\frac{Q_p(1)}{1+\pi r_c^2\delta(r_w,\Delta t,1)} \tag{31}$$

Assuming $Q_a(0)$ to be very near to zero,

$$Q_a(1)=\frac{2Q_p(1)}{1+\pi r_c^2\delta(r_w,\Delta t,1)} \tag{32}$$

$\bar{Q}_a(n), n=1,...,N$ are solved in succession. After solving $\bar{Q}_a(n)$, for n=1, 2,…,N, the drawdown in the aquifer at any distance r is found generating the corresponding kernel coefficients $\delta(r,\Delta t,N)\left[=\frac{1}{\Delta t}\{U(r,N\Delta t)-U(r,(N-1)\Delta t)\}\right]$ and applying the convolution technique.

Having solved the direct problem, the inverse problem is solved next.

SOLUTION TO THE INVERSE PROBLEM

Let T^* and ϕ^* be approximate values differing by ΔT and $\Delta\phi$ from the true transmissivity and storage coefficient of the confined homogeneous and isotropic aquifer which was at rest prior to the aquifer test. For solving the inverse problem, the objective function to be minimized is sum of the squares of the error, i.e., squares of the differences in observed drawdowns and predicted drawdowns corresponding to T^* and ϕ^*.

$$\underset{\Delta T,\,\Delta\phi}{Min}\left\{\sum_{i=1}^{N}\left\{s_0(i)-s_c\left(t_i,T^*+\Delta t,\phi^*+\Delta\phi\right)\right\}^2\right\} \tag{33}$$

An initial guess is made for T^*,ϕ^* and ΔT and $\Delta\phi$ are solved through minimizing the error.

The Taylor series expansion of $s_c\left(t_i,T,\phi\right)$ at $T=T^*$ and $\phi=\phi^*$, and neglecting higher order terms,

$$s_c\left(t_i,T^*+\Delta T,\phi^*+\Delta\phi\right)=s_c\left(t_i,T,\phi\right)\Big|_{T^*,\phi^*}+\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}\Delta T+\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial\phi}\right|_{T^*,\phi^*}\Delta\phi \tag{34a}$$

The partial derivatives are to be determined numerically as follows:

$$\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}=\frac{s_c\left(t_i,T^*+\varepsilon_1,\phi^*\right)-s_c\left(t_i,T^*,\phi^*\right)}{\varepsilon_1} \tag{34b}$$

$$\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial\phi}\right|_{T^*,\phi^*}=\frac{s_c\left(t_i,T^*,\phi^*+\varepsilon_2\right)-s_c\left(t_i,T^*,\phi^*\right)}{\varepsilon_2} \tag{34c}$$

where $\varepsilon_1,\varepsilon_2$ are small increments in transmissivity and storage coefficient.

Incorporating $s_c\left(t_i,T^*+\Delta T,\phi^*+\Delta\phi\right)$ i.e. equation (34a), in equation (33), the minimization problem reduces to

$$\underset{\Delta T,\,\Delta\phi}{Min}\left[\sum_{i=1}^{N}\left\{s_0(i)-\left(s_c\left(t_i,T,\phi\right)\Big|_{T^*,\phi^*}+\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}\Delta T+\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial\phi}\right|_{T^*,\phi^*}\Delta\phi\right)\right\}^2\right] \tag{35}$$

Equating the partial derivative of the above objective function with respect to ΔT with zero

$$\left[\sum_{i=1}^{N}-2\left\{s_0(i)-\left(s_c\left(t_i,T,\phi\right)\Big|_{T^*,\phi^*}+\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}\Delta T+\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial\phi}\right|_{T^*,\phi^*}\Delta\phi\right)\right\}\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}\right]=0 \tag{36}$$

Simplifying

$$\left[\sum_{i=1}^{N}\left\{\left(\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}\right)^2\right\}\right]\Delta T$$
$$+\left[\sum_{i=1}^{N}\left\{\left(\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial\phi}\right|_{T^*,\phi^*}\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}\right)\right\}\right]\Delta\phi$$
$$=\sum_{i=1}^{N}\left\{s_0(i)\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}\right\}-\sum_{i=1}^{N}\left\{s_c\left(t_i,T,\phi\right)\Big|_{T^*,\phi^*}\left.\frac{\partial s_c\left(t_i,T,\phi\right)}{\partial T}\right|_{T^*,\phi^*}\right\} \tag{37a}$$

or

$$a(1,1)\Delta T + a(1,2)\Delta\phi = c(1) \tag{37b}$$

where

$$a(1,1) = \sum_{i=1}^{N}\left\{\left(\left.\frac{\partial s_c(t_i,T,\phi)}{\partial T}\right|_{T^*,\phi^*}\right)^2\right\};\ \mathrm{a}(1,2) = \sum_{i=1}^{N}\left\{\left(\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\left.\frac{\partial s_c(t_i,T,\phi)}{\partial T}\right|_{T^*,\phi^*}\right)\right\} \text{ and}$$

$$c(1) = \sum_{i=1}^{N}\left\{s_0(i)\left.\frac{\partial s_c(t_i,T,\phi)}{\partial T}\right|_{T^*,\phi^*}\right\} - \sum_{i=1}^{N}\left\{\left.s_c(t_i,T,\phi)\right|_{T^*,\phi^*}\left.\frac{\partial s_c(t_i,T,\phi)}{\partial T}\right|_{T^*,\phi^*}\right\}$$

Similarly, equating the partial derivative of the objective function with respect to $\Delta\phi$ with zero

$$\left[\sum_{i=1}^{N}-2\left\{s_0(i)-\left(\left.s_c(t_i,T,\phi)\right|_{T^*,\phi^*}+\left.\frac{\partial s_c(t_i,T,\phi)}{\partial T}\right|_{T^*,\phi^*}\Delta T+\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\Delta\phi\right)\right\}\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\right]=0 \tag{38}$$

Simplifying

$$\left[\sum_{i=1}^{N}\left\{\left(\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\left.\frac{\partial s_c(t_i,T,\phi)}{\partial T}\right|_{T^*,\phi^*}\right)\right\}\right]\Delta T + \left[\sum_{i=1}^{N}\left\{\left(\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\right)^2\right\}\right]\Delta\phi$$

$$= \sum_{i=1}^{N}\left\{s_0(i)\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\right\} - \sum_{i=1}^{N}\left\{\left.s_c(t_i,T,\phi)\right|_{T^*,\phi^*}\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\right\} \tag{39a}$$

or

$$a(2,1)\Delta T + a(2,2)\Delta\phi = c(2) \tag{39b}$$

where

$$a(2,1) = \sum_{i=1}^{N}\left\{\left(\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\left.\frac{\partial s_c(t_i,T,\phi)}{\partial T}\right|_{T^*,\phi^*}\right)\right\};\ a(2,2) = \sum_{i=1}^{N}\left\{\left(\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\right)^2\right\} \text{ and}$$

$$c(2) = \sum_{i=1}^{N}\left\{s_0(i)\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\right\} - \sum_{i=1}^{N}\left\{\left.s_c(t_i,T,\phi)\right|_{T^*,\phi^*}\left.\frac{\partial s_c(t_i,T,\phi)}{\partial \phi}\right|_{T^*,\phi^*}\right\}$$

The unknown ΔT and $\Delta\phi$ are solved from the algebraic equations (37b) and (39b). Solving for $\Delta\phi$ and ΔT we get

$$\Delta\phi = \frac{\dfrac{c(1)}{a(1,1)} - \dfrac{c(2)}{a(2,1)}}{\dfrac{a(1,2)}{a(1,1)} - \dfrac{a(2,2)}{a(2,1)}} \tag{40}$$

and

$$\Delta T = \frac{c(1)}{a(1,1)} - \frac{a(1,2)}{a(1,1)}\Delta\phi \tag{41}$$

An Example

A set of synthetic observation data generated using $T = 0.015 m^2/\min$ and storage coefficient $\phi = 0.3$ are as given in Table 6. Predict the T, ϕ making an initial guess $T^* = 0.01 m^2/\min$ and $\phi^* = 0.2$. The drawdown is measured in the large diameter well of radius 1.725m. The pumping rate is $Q = 0.225 m^3/\min$.

Table 6: Synthetic Drawdown Data

Time of observation (min)	Observed Drawdown(m)	Time of observation (min)	Observed Drawdown(m)
2	4.39E-02	30	5.12E-01
3	6.47E-02	35	5.81E-01
4	8.50E-02	40	6.47E-01
5	1.05E-01	45	7.10E-01
6	1.24E-01	50	7.70E-01
7	1.43E-01	60	8.85E-01
8	1.62E-01	70	9.92E-01
9	1.80E-01	80	1.09E+00
10	1.98E-01	90	1.19E+00
12	2.33E-01	100	1.28E+00
14	2.67E-01	120	1.44E+00
16	3.00E-01	140	1.59E+00
18	3.33E-01	160	1.73E+00
20	3.64E-01	180	1.86E+00
25	4.40E-01	200	1.98E+00

Table 7: Convergence of T^* and ϕ^* with Successive Iteration

Iteration no	T^*	ϕ^*	ΔT	$\Delta\phi$	C(1)	C(2)
1	0.01	0.2	0.002018	0.160276	0.00E+00	2.32E-09
2	0.012018	0.360276	0.002534	-0.05646	0.00E+00	-6.99E-09
3	0.014552	0.303812	0.000435	-0.00361	8.88E-16	3.81E-10
4	0.014987	0.300203	0.000012	-0.00018	0.00E+00	-2.10E-11
5	0.014999	0.300024	0.000001	-2.2E-05	1.73E-18	-2.50E-12
6	0.015	0.300002	0	-2E-06	0.00E+00	5.66E-13

Thus using synthetic drawdown data in a large-diameter well, we have checked that Marquardt Algorithm successfully predicts the true transmissivity and storage coefficient starting with an initial guess different from the true value.

Determination of Aquifer Parameters Applying Marquardt Algorithm to Observed Drawdown in a Well during Pumping [the aquifer test is conducted in a large diameter well (Baori)]

Location: in Jodhpur at Subhash Chowk; Pumping Rate, Q = 0.225 $m^3/\min$.

Radius of the Well = 1.725m

Table 8 Observed drawdown

Time of observation (min)	Observed Drawdown(m)	Time of observation (min)	Observed Drawdown(m)
2	4.00E-02	30	4.45E-01
3	5.50E-02	35	5.10E-01
4	7.00E-02	40	5.70E-01
5	8.50E-02	45	6.35E-01
6	1.00E-01	50	7.00E-01
7	1.15E-01	60	8.00E-01
8	1.30E-01	70	9.80E-01
9	1.45E-01	80	1.03E+00
10	1.60E-01	90	1.13E+00
12	1.90E-01	100	1.24E+00
14	2.25E-01	120	1.43E+00
16	2.45E-01	140	1.58E+00
18	2.75E-01	160	1.71E+00
20	3.00E-01	180	1.83E+00
25	4.00E-01	200	1.97E+00

Table 9 Transmissivity and Storage Coefficient as Obtained through Successive Iteration

Iteration no	T^*	ϕ^*	ΔT	$\Delta\phi$	C(1)	C(2)
1	0.024	0.01	-0.0518	0.117059	5.70E-06	3.44E-06
2	0.024	0.127059	-0.03277	0.418365	-7.69E-06	-6.00E-07
3	0.024	0.127059	-0.03485	0.423725	-7.69E-06	-6.00E-07
4	0.024	0.127059	-0.03485	0.423725	-7.69E-06	-6.00E-07

Iterated Transmissivity, T= 0 .024 m^2/min =24.56 m^2/day

Iterated Storage Coefficient, ϕ = 0.127

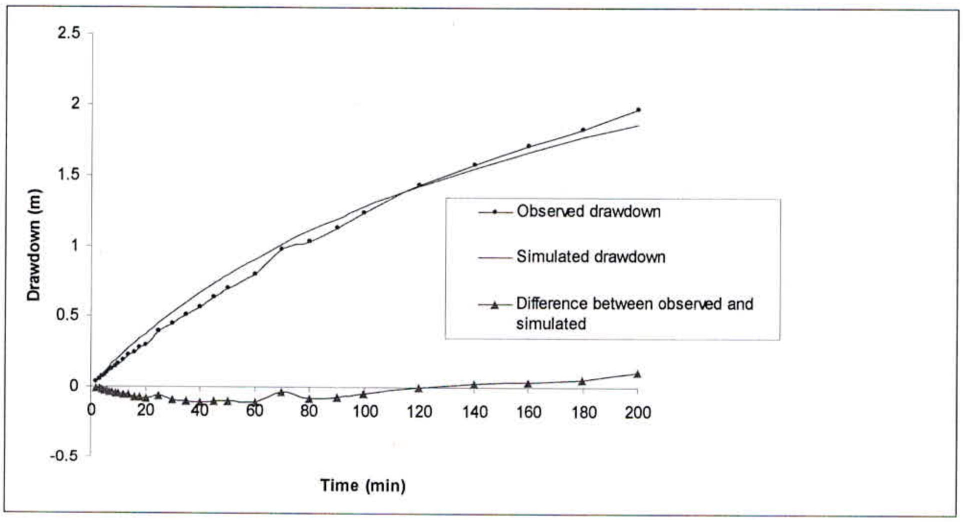

Fig. 2 Observed and Simulated drawdowns for T = 0 .0347 m^2/min =49.97 m^2/day and Storage Coefficient, ϕ = 0.046 with Marquardt Algorith

Table 10 Comparison of Observed Drawdowns with Simulated Drawdowns for T= 0 .024 m^2 / min =24.56 m^2 / day and Storage Coefficient, ϕ = 0.127

Time(min)	Observed drawdown(m)	Simulated drawdown(m)	Time(min)	Observed drawdown(m)	Simulated drawdown(m)
2	0.04	0.0444	30	0.445	0.5196
3	0.055	0.0656	35	0.51	0.5884
4	0.07	0.0862	40	0.57	0.6539
5	0.085	0.1063	45	0.635	0.7164
6	0.1	0.126	50	0.7	0.7763
7	0.115	0.1453	60	0.8	0.8887
8	0.13	0.1643	70	0.98	0.9926
9	0.145	0.1829	80	1.03	1.0892
10	0.16	0.2012	90	1.13	1.1793
12	0.19	0.237	100	1.235	1.2638
14	0.225	0.2717	120	1.43	1.418
16	0.245	0.3054	140	1.58	1.5557
18	0.275	0.3383	160	1.71	1.6798
20	0.3	0.3703	180	1.83	1.7924
25	0.4	0.447	200	1.97	1.8952

Appendix I
Discrete Kernel, $\delta(r_w, \Delta t, N)$

Hantush(1964) has derived the well function for computation of drawdown in an artesian aquifer due to pumping from a fully penetrating well of finite radius starting from the basic solution given by Carslaw and Jaeger (1959) for an analogous heat conduction problem. Let the unit step response function for piezometric rise at the well face of a fully penetrating recharge well and a confined aquifer system be designated as $U(r_w, I\Delta t)$. According to Hantush (1964) it is given by:

$$U(r_w, N\Delta t) = \frac{1}{4\pi T}\left[\frac{4}{\pi}\int_0^\infty \{1-\exp(-\tau x^2)\}\, f_1(x)dx\right] \quad (1)$$

in which,

$$\tau = \frac{Tt}{\varphi r_w^{\,2}};\ t = N\Delta t; \qquad f_1(x) = \frac{J_1(x)Y_0(\rho x) - J_0(\rho x)Y_1(x)}{x^2\left[J_1^{\,2}(x) + Y_1^{\,2}(x)\right]}; \quad \rho = \frac{r}{r_w} = 1;$$

$J_0(x)$, $J_1(x)$ =Bessel functions of first kind of zero and first order respectively; $Y_0(x)$ $Y_1(x)$= Bessel functions of second kind of zero and first order respectively; T= transmissivity (m^2/day), and ϕ=storativity of the aquifer; r_w = radius of the well or shaft(m).

The integral in (1) is an improper integral as the upper limit of integration is infinite. The improper integral is reduced to a proper integral as described below.

$$I = \int_0^\infty \left[1 - exp(-\tau x^2)\right] f_1(x)dx$$

$$= \int_0^1 \left[1 - exp(-\tau x^2)\right] f_1(x)dx + \int_1^\infty \left[1 - exp(-\tau x^2)\right] f_1(x)dx$$

$$= I_1 + I_2$$

$$I_1 = \int_0^1 \left[1 - exp(-\tau x^2)\right] f_1(x)dx = 0.5 \int_{-1}^1 \left[1 - exp\{\frac{-\tau(1+v)^2}{4}\}\right] f_1(\frac{1+v}{2})dv$$

Expanding the exponential term, and applying L' Hospital's rule, it can be shown that as v tends to -1, the integrand tends to 0. The integral I_1 is a proper integral and can be evaluated numerically using Gauss qudrature.

$$I_2 = \int_1^\infty \left[1 - exp(-\tau x^2)\right] f_1(x)dx = \int_0^1 \left[1 - exp(-\tau / v^2)\right] f_1(1/v)\frac{dv}{v^2}$$

$$= 0.5 \int_{-1}^1 \left[1 - \exp\left\{\frac{-4\tau}{(1+y)^2}\right\}\right] f_1(\frac{2}{1+y})\frac{4dy}{(1+y)^2}$$

Limit of the integrand at the lower is found as described below.

As $y \to -1$, $\left[1 - exp\left\{\frac{-4\tau}{(1+y)^2}\right\}\right] \to 1$

$$[\frac{4}{(1+y)^2}] f_1(\frac{2}{1+y}) = \left[\frac{4}{(1+y)^2}\right] \frac{J_1(\frac{2}{1+y})Y_0(\rho\frac{2}{1+y}) - J_0(\rho\frac{2}{1+y})Y_1(\frac{2}{1+y})}{(\frac{2}{1+y})^2\left[J_1^2(\frac{2}{1+y}) + Y_1^2(\frac{2}{1+y})\right]}$$

$$= \frac{J_1(\frac{2}{1+y})Y_0(\rho\frac{2}{1+y}) - J_0(\rho\frac{2}{1+y})Y_1(\frac{2}{1+y})}{\left[J_1^2(\frac{2}{1+y}) + Y_1^2(\frac{2}{1+y})\right]}$$

$$= \frac{J_1(\frac{2}{1+y})Y_0(\rho\frac{2}{1+y})}{\left[J_1^2(\frac{2}{1+y}) + Y_1^2(\frac{2}{1+y})\right]} - \frac{J_0(\rho\frac{2}{1+y})Y_1(\frac{2}{1+y})}{\left[J_1^2(\frac{2}{1+y}) + Y_1^2(\frac{2}{1+y})\right]}$$

As $y \to -1$, $Y_1(\frac{2}{1+y}) \to 0$; hence,

$$\frac{J_1(\frac{2}{1+y})Y_0(\rho\frac{2}{1+y})}{\left[J_1^2(\frac{2}{1+y})+Y_1^2(\frac{2}{1+y})\right]}$$

$$\cong\frac{J_1(\frac{2}{1+y})Y_0(\rho\frac{2}{1+y})}{J_1^2(\frac{2}{1+y})}=\frac{Y_0(\rho\frac{2}{1+y})}{J_1(\frac{2}{1+y})}\frac{\sqrt{\frac{(1+y)}{\rho\pi}}\;sin\left(\frac{2\rho}{1+y}-\frac{\pi}{4}\right)}{\sqrt{\frac{(1+y)}{\pi}}\;cos\left(\frac{2}{1+y}-\frac{3\pi}{4}\right)}$$

=1 (since ρ=1)

Similarly,

$$\frac{J_0(\rho\frac{2}{1+y})Y_1(\frac{2}{1+y})}{\left[J_1^2(\frac{2}{1+y})+Y_1^2(\frac{2}{1+y})\right]}\to 1$$

Thus I_2 can be evaluated using Gauss qudrature.

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