Finite Element Solution of Forchheimer Flows Around Fully Penetrating Wells With Storage in Confined Aquife~s

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ABSTRACT

Most of the wells in Asian countries are dug wells of large diameter with huge amounts of storage in them. The application of existing methods either for modelling the aquifer or for simulating the drawdown history would be erroneous in such cases. Hence a study of transient flow towards finite diameter wells with storage has been attempted, with a view to develop type curves required for the identification of aquifer parameters.

This paper deals with solution for fully screened wells in confined aquifers considering the flow towards the wells to be non-linear and governed by Forchheimer law. A digital model based on the finite element technique was developed and analysed for the case of wells with storage. An attempt has been made to generalise the solution and type curves have been presented at discrete distances from the discharging well for the different storage parameters and for parameters identified to define the non-linear flow. From the critical observations of these type curves, ' storage affected zone ' is demarcated in spatial and temporal coordinates. The concept of ' time of deviation ' and ' time of merging ' is introduced and the affect of Forchheimer parameters has been studied. A simple method has been suggested to determine the transmissivity of the aquifer from the early time record of pumping history.

INTRODUCTION

The formulation of strategy for the optimal management of the ground water demands a knowledge of the relationship between the pumpages in the well to the drawdowns in the aquifer in spatial and temporal coordinates. Though, there are several such relationships available in well hydraulics, most of them consider the discharging well to be a line sink, and do not consider the storage in the well. Also, the validity of the Darcy law has been taken granted even in the close vicinity of the well. However, most of the wells in India are dug wells of large diameter with huge amounts of storage in them and are used mainly for irrigation purposes where huge amounts of water is drawn from them, causing heavy drawdowns. In such cases, the validity of linear Darcy law is highly questionable. Hence, a study of transient flow towards finite diameter wells with storage govërened by the Forchheimer law has been attempted, with a view to develop type curves required for the identification of aquifer parameters. In the present investigation, a digital model based on the finite element technique is developed and used to obtain the solutions for fully screened wells with storage in confined aquifers. An attempt has been made to generalise these solutions and type curves have been presented at discrete distances from the discharging well.

FORCHHEIMER EQUATION

The limited validity of Darcy's law has led to the suggestion of relationship that would be true over all the flow ranges encountered. Forchheimer (1901) proposed an equation of the form

$$i = aV + bV^2 \tag{1}$$

where

i = absolute hydraulic gradient,

V = absolute macroscopic velocity, and

a, b = Forchheimer Coefficients signifying the viscous and inertial terms respectively.

When the absolute macroscopic velocity is sufficiently small, equation 1 is reduced to

$$i = aV \tag{1a}$$

where

$a = 1/K$, K being hydraulic conductivity

Hence, the validity of the equation 1 is assured over the entire flow domain. Though, there are many other forms of relationships to describe the non-linear law, it is generally agreed upon that the form suggested by equation 1 is the most appropriate. Ergan et al (1949), Engelund(1953), Ahmed et al (1969) and many others adopted the Forchheimer equation to describe the non-Darcy flow.

MATHEMATICAL MODEL

A brief description of the flow field and the assumptions involved in idealising the same are given below:

A diagramatic sketch of a fully screened well with storage in a confined aquifer is shown in Fig.1. The aquifer of thickness 'm' is considered to be of infinite radial extent with the well at its centre, such that all physical conditions are symmetrical with respect to the axis of the well. The aquifer is homogeneous, isotropic and has a negligible dip. It is overlain and underlain by aquicludes. The well is screened fully and is pumped at a constant rate 'Q'. The equilibrium conditions are yet to reach and flow is considered to be in transient state. The radius of the well is denoted by ' r_w ' and that of the well casing by ' r_c '. At any instant of time ' t ', (total time from the starting of the pump) the drawdown in the well is designated by ' s_w ' . The height of the nonpumping piezometric surface is indicated by ' h_o'. For the purpose of discretization, the infinite aquifer is replaced by a finite system with inflow potential boundary located at $r = r_o$ from the Z- axis. At any radial distance 'r ' the nieght of piezometric surface is designated by ' h ' and the drawdown by ' s '.

a) Governing field equation for transient flow towards wells:.
For the regime of non-Darcy flow, the Forchheimer velocity relationship is re-written as

$$V = E \left| \frac{\partial h}{\partial l} \right| \qquad (2)$$

where E is the coefficient of effective hydraulic conductivity given by

$$E = \left[\frac{a}{2} + \left(\frac{a^2}{4} + b \left| \frac{\partial h}{\partial l} \right| \right)^{1/2} \right]^{-1} \qquad (3)$$

where

$$\left(\frac{\partial h}{\partial l} \right)^2 = \left(\frac{\partial h}{\partial r} \right)^2 + \left(\frac{\partial h}{\partial z} \right)^2$$

The general equation for non-Darcy flow thus becomes

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r.E. \frac{\partial h}{\partial r} \right) + \frac{\partial}{\partial z} \left(E. \frac{\partial h}{\partial z} \right) = S_s \frac{\partial h}{\partial t} \qquad (4)$$

Since the expression for E is a dependent variable over h the partial differential equation for non-Darcy flow (equation 4) is non-linear. Hence, many of the analytical techniques for the solution of linear partial differential equations, do not hold good. Thus, in the present analysis an iterative approach is proposed for minimising the functional obtained from the equation 4.

b) Boundary conditions for a well with storage in confined aquifer.
The boundary conditions for the problem (Fig.1).

CD	$z = m$	$r_w \leq r \leq r_o$	$\frac{\partial h}{\partial z} = 0$	(5)
DE	$r = r_o$	$0 \leq z \leq m$	$h = h_o$	(6)
BA	$z = 0$	$r_w \leq r \leq r_o$	$\frac{\partial h}{\partial z} = 0$	(7)
AC	$r = r_c$	$0 \leq z \leq m$	The discharge prescribed condition.	

$$\text{i.e.} \quad -2\pi r_w \int_0^m k \left.\frac{\partial h}{\partial r}\right|_{r=r_w} dz + \pi r_c^2 \left.\frac{\partial h}{\partial t}\right|_{r=r_w} \tag{8}$$

c) Initial condition for these problems is

$$h\ (r,\ o\ ) = h_o \tag{9}$$

VARIATIONAL FORMULATION

Variational form of the equation 4 may be obtained by considering an equivalent variational problem and adopting Euler-Lagraunge equation from the calculus of variations. The functional describing the non-Darcy flow through isotropic aquifers is obtained by comparing the terms of Euler-Lagraunge equation with those of equation 4, integrating them and summing them subsequently. Thus, the functional over the region 'R' is

$$\left[I\ (h)\right]_R = \int_t^{t+\Delta t} \int_R F.dR.dt \tag{10}$$

where

$$F = -\frac{a}{2b}\left(\left|\frac{\partial h}{\partial l}\right|\right) + \left[\left(\frac{a}{2b}\right)^2 + \left|\frac{\partial h}{\partial l}\right| / b\right]^{3/2} \cdot \frac{2}{3}\, b$$

$$+ S_s \cdot h \cdot \frac{\partial h}{\partial t} \tag{11}$$

and $dR = 2\pi r.\ dr.\ dz$

Now the variational problem reduces to finding an admissible function that minimises the functional and also satisfies the existing initial and boundary conditions of the system. Accordingly, to satisfy the prescribed discharge condition the following term has to be added to the equation 11.

$$\left[I\ (h)\right]_B = \int_t^{t+\Delta t} \int_B h\ \bar{q}\ dB\ dt \tag{12}$$

where $\bar{q}$ is the prescribed flux on the boundary B and Δt is the time increment.

The functional thus obtained over the entire flow Region $\bar{R}$ is given by

$$\Big[I \quad (h) \Big]_{\bar{R}} = \int_t^{t+\Delta t} \int_R \left\{ \frac{-a}{2b} \left| \frac{\partial h}{\partial l} \right| + \frac{2}{3} b \left[\left(\frac{a}{2b} \right)^2 + \left| \frac{\partial h}{\partial l} \right| / b \right]^{3/2} + S_s \, h \frac{\partial h}{\partial t} \right\} dR . dt + \int_t^{t+\Delta t} \int_B h . \bar{q} . \, dB . dt \qquad (13)$$

FINITE ELEMENT SOLUTION

Discretization of the Flow System

An approximate minimization of the functional is achieved by the 'finite element method', which may be viewed as an extended application of the Raleigh-Ritz method over the subregions of the flow system. The aquifer is discretized into an assemblage of rectangular ring elements with a fine mesh near the well and a coarse one far away. A transition zone connects the fine mesh and the coarse mesh and it is discretized by traingular elements. A progressively increasing width is adopted for the elements using $\Delta r_i = (1.3) \Delta r_{i-1}$.

Variation of the head is assumed to be bilinear in each rectangular element. The head in an individual element may be expressed as, (with summation over the repeated subscript implied)

$$h = N_i \ (r, z) \ h_i \ (t) \qquad (14)$$

where i (= 1,2,3,4) indicates the local nodal number for the element. $N_i(r, z)$ are piecewisely defined functions of the coordinates, called shape function and $h_i(t)$ are the values of the nodal heads at time t. The expressions for N_i can be easily derived.

Element Matrices

Substituting equation 14 in equation 13, the functional for an element $I^e(h)$ is obtained, which depends upon the nodal heads h_i only. Evaluating $\partial I^e / \partial h_i$, and denoting

$$\left[\frac{\partial I^e}{\partial h} \right]^T = \left[\frac{\partial I^e}{\partial h_1} \quad \frac{\partial I^e}{\partial h_2} \quad \frac{\partial I^e}{\partial h_3} \quad \frac{\partial I^e}{\partial h_4} \right] \qquad (15)$$

one obtains

$$\frac{\partial I^e}{\partial h} = \int_t^{t+\Delta t} \left\{ \left[C^e\right]\left[h^e\right] + \left[P^e\right]\left[\frac{\partial h^e}{\partial t}\right] + \left[F^e\right] \right\} dt \quad (16)$$

where

$$\left[h^e\right]^T = \left[h_1 \;\; h_2 \;\; h_3 \;\; h_4\right] \quad (17)$$

$$\left[C^e\right] = \int_{R^e} K \left[S\right]^T \left[S\right] \, dR \quad (18)$$

$$\left[P^e\right] = \int_{R^e} S_s \left[N\right]^T \left[N\right] \, dR \quad (19)$$

$$\left[F^e\right] = \int_{B^e} \bar{q} \left[N\right]^T \, dB \quad (20)$$

with

R^e = region of the element;

B^e = portion of the boundary B belonging to the element (if any), and

$$\left[N\right] = \left[N_1 \;\; N_2 \;\; N_3 \;\; N_4\right] \quad (21)$$

$$\left[S\right] = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} & \frac{\partial N_4}{\partial r} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_4}{\partial z} \end{bmatrix} \quad (22)$$

The matrices $\left[C^e\right]$, $\left[P^e\right]$ and $\left[\;^e\right]$ are respectively called the conductance matrix, the stotativi matrix and the column matrix of nodal discharges for the eleme. The elements of $\left[F^e\right]$ would be zero for all the nodes except for t , flux- prescribed boundary nodes.

Gross Matrices

The elemental contributions I^e would be added to obtain I for the entire flow domain and the minimization of I requires,

$$\frac{\partial I}{\partial h_i} = \sum_{e=1}^{M} \left(\frac{\partial I^e}{\partial h_i}\right) = 0 \; ; \; i = 1,2,\ldots\ldots n \quad (23)$$

where n = total number of nodes and M = total number of elements. Equation 23 can be expressed as

$$\int_{t}^{t+\Delta t} \left\{ [C][h] + [P]\left[\frac{\partial h}{\partial t}\right] + [F] \right\} dt = 0 \qquad (24)$$

where the gross matrices $[C]$, $[P]$, $[F]$ for the flow system are assembled by the summation of the corresponding individual element matrices.

Integration in time Domain

The solution is advanced in time using Crank - Nicolson scheme which gives

$$[D]\,[h]^{t+\Delta t} = [G]\,[h] - \frac{\Delta t}{2}\left[[F]^{t} + [F]^{t+\Delta t}\right] \qquad (25)$$

where

$$[D] = [P] + \frac{\Delta t}{2}\,[C] \qquad (26)$$

$$[G] = [P] - \frac{\Delta t}{2}\,[C] \qquad (27)$$

The symmetric and banded nature of $[C]$ and $[P]$ is exploited by storing only the nonzero upper diagonal elements in a rectangular array. The successive time steps Δt_i were chosen to increase geometrically, such that $\Delta t_i = 1.4 \times \Delta t_{i-1}$. However, care was taken to ensure that numerical oscillations are not introduced in the Crank -Nicolson scheme, even for the largest time step used.

Prescribed Flux Boundary Conditions

The well is pumped at a constant, flow rate; but the discharge drawn from the aquifer is unknown, due to the storage in the well. Hence, the implementation of the boundary condition given by equation 8 requires a trial and error approach. The head in the well at any time is assumed and is corrected iteratively till equation 8 is satisfied in the limits of the prescribed accuracy. For the first two time steps the trial guess values are used for the head and for subsequent time steps, logarithmic extrapolation over time is used for the initial guess value. With this head prescrib ed condition in the well, the equation

25 is solved and the total discharge from the aquifer and well storage is compared with the prescribed discharge, untill convergence, within certain prescribed limits, is obtained. Convergence is obtained in three iterations at early times and in two iterations at late times, when the effects of well storage diminish.

In any particular element, if $bv/a > 0.01$, Forchheimer relation is used. Otherwise, the flow is considered to be governed by Darcy law by neglecting the second term in equation 4. The equation 25 is non-linear, since the effective hydraulic conductivity, E is a function of the nodal head distribution. So, an iterative procedure with an over-relaxation factor of 1.6 is used for the solution. In general, convergence was obtained in about 3 to 4 interations.

Results of Analysis and Conclusions

The functional relationship between the dimensionless drawdown $s/(Q/4\pi T)$ and the dimensionless time $1/u$, is given by

$$\frac{s}{(Q/4\pi T)} = W\left(\frac{1}{u}, \frac{r}{r_w}, \beta, \lambda_w, \xi\right) \qquad (28)$$

T = transmissivity of the aquifer (= k m)

S = Storativity of the aquifer ($= S_s$ m)

$u = r^2 S/(4T t)$

V_{cr} = Velocity below which Darcy's law is applicable

$\beta = r_c^2 / (r_w^2 S)$

$\lambda_w = b\, v_w/a$

$v_w = Q/(2\pi m\, r_w)$

$\xi = b\, v_{cr}/a$

The storage parameter defined by Papadopulos and Cooper(1967) gives infinite values for wells with no storage; and to remove this incongruity, its reciporcocal, β is adopted here.

Finite element model has been used to obtain the type curves for some discrete values of the different parameters viz., r/r_w = 1,8,16,48,64, 160; $\beta = 10^2, 10^3, 10^4, 10^5$; λ_w = 3,6,12 and ξ = 0.01. Three type curves for r/r_w = 1 are presented in figures 2,3,4 for different values of λ_w.

On all the type curves three distinct regions can be identified. The region I refers to the straight line portion corresponding to the early time drawdown history, when the discharge is drawn almost entirely from the storage in the well. The region II refers to the portion of the curve from the point at which it deviates from the straight line to the point at which it merges with the curves for the well with no storage. The region III refers to the portion of the curve which merges with the curve $\beta = 0$. In this region, the effect of well storage becomes negligible and the discharge is obtained almost entirely from the aquifer storage. These three regions are noted on the type curves generated for all combinations of the parameters.

Time of Deviation

For the type curves for $r/r_w = 1$, i.e., at the well face, it is possible to locate a deviation point which marks the junction of regions I and II. Designating the corresponding values of u and t as u_{wd} and t_{wd} respectively, one can observe that

$$u_{wd}\ \beta = C'_d \tag{29}$$

where C'_d is a constant. Substituting $u_d = r_w^2\ S/(4Tt_d)$ and with β as defined, one obtains

$$t_d = C_d\ r_c^2\ /T \tag{30}$$

C_d is seen to increase with λ_w and the value of T to be used in the equation 30 is the one corresponding to Darcy flow away from the well. The time of deviation increases due to reduction in the effective transmissivity of the region experiencing Forchheimer flow in th e neighbourhood of the well.

If one locates the point of tangency on the time-drawdown curves for the pumping well, in a field test, equation 30 can give the value of T for the aquifer in a simple way.

Time of merging

The junction of regions II and III may be called a 'merging point' and the corresponding values of u and t may be designated as u_m and t_m respectively. It is noted from the type curves that although u_m varies with β, the product $u_m\ \beta$ remains approximately constant other parameters remaining the same. Thus,

$$u_m \beta = C'_m \tag{31}$$

where,

C'_m is a constant. Substituting $u_m = r^2 S(4 T t_m)$ and with β as defined, one obtains

$$t_m = C_m r_c^2 / T \tag{32}$$

where $$C_m = (r/r_w)^2 / (4 u_m \beta) \tag{33}$$

For all $t > t_m$, the effects of well storage can be neglected. For a given r/r_w, t_m increases with the radius of the casing well and decreases with the transmissivity of the aquifer. C_m depends upon r/r_w, λ_w, ξ and can be taken to be independent of β. Other parameters being same, C_m would increase with r/r_w. C_m is more in the case of non-linear flow than when Darcy flow is considered. This can be attributed to a reduction in the effective transmissivity of the region experiencing non-linear flow. The storage in the well reduces the drawdowns for all $t > t_m$. The effects of storage in the well reduce the drawdowns. Forchheimer flows involve larger drawdowns than the Darcy flows and so counterac-t the effects of well storage in reducing the drawdowns. The effects of non-linear flow would be felt for a short distance away from the well, while the effects of storage in the well would extend far into the interior of the aquifer.

The time at which the effect of well storage on the drawdown becomes insignificant can be obtained from the type curves presented. For all observations taken before such time, the type curves considering the well storage have to be used in the estimation of the aquifer storativity and transmissivity. These parameters may be roughly evaluated by the curve matching methods. Using them as initial guess values, the finite element models developed can be implemented on a digital computer. The drawdown history, so simulated, can be compared to the observed drawdown history and the parameters can be adjusted by an iterative procedure to minimize the deviations between the observed and the simulated drawdowns, based on an appropriate error norm. This leads one to a special problem of parameter estimation.

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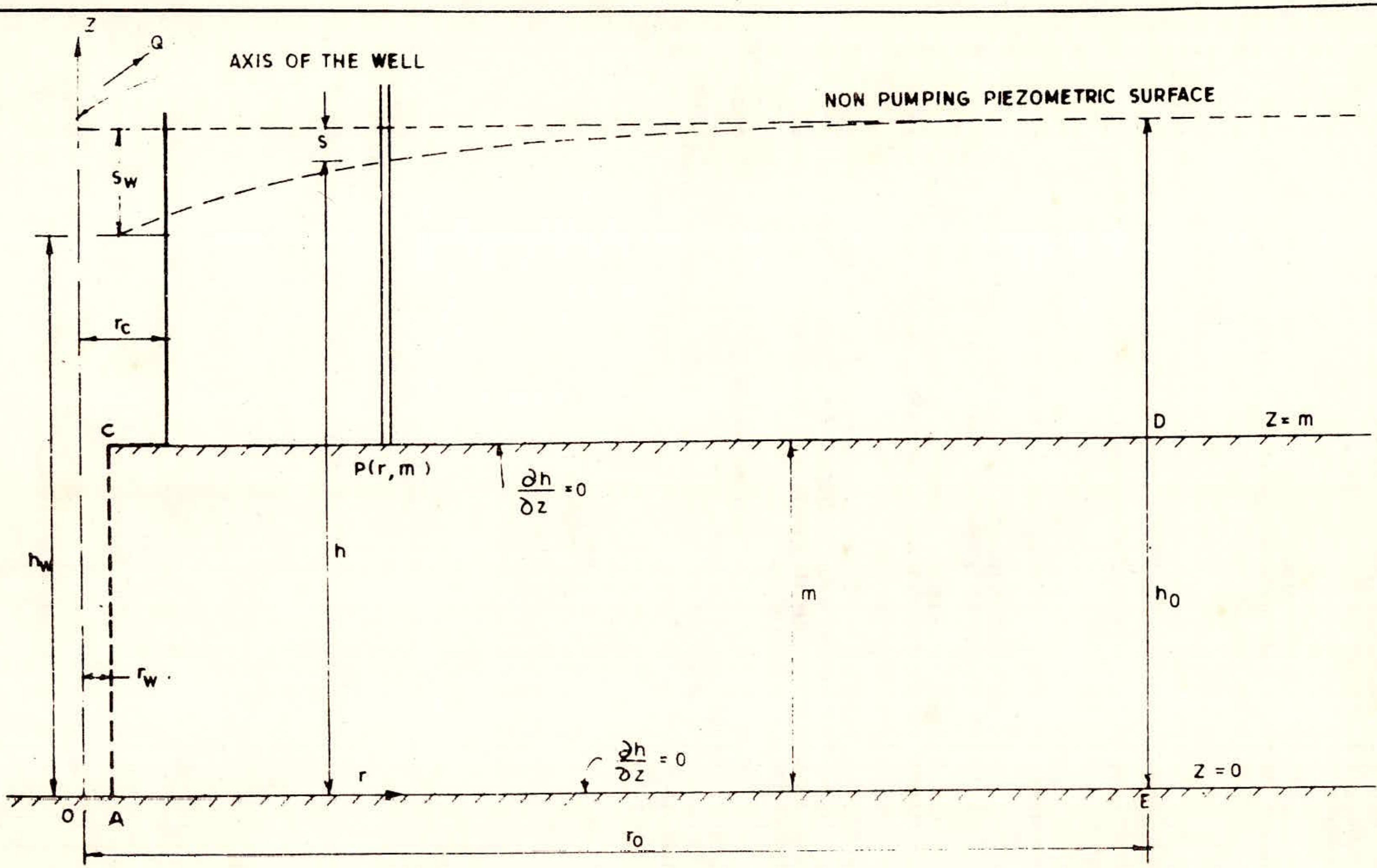

FIG.1_FULLY SCREENED WELL WITH STORAGE IN A CONFINED AQUIFER

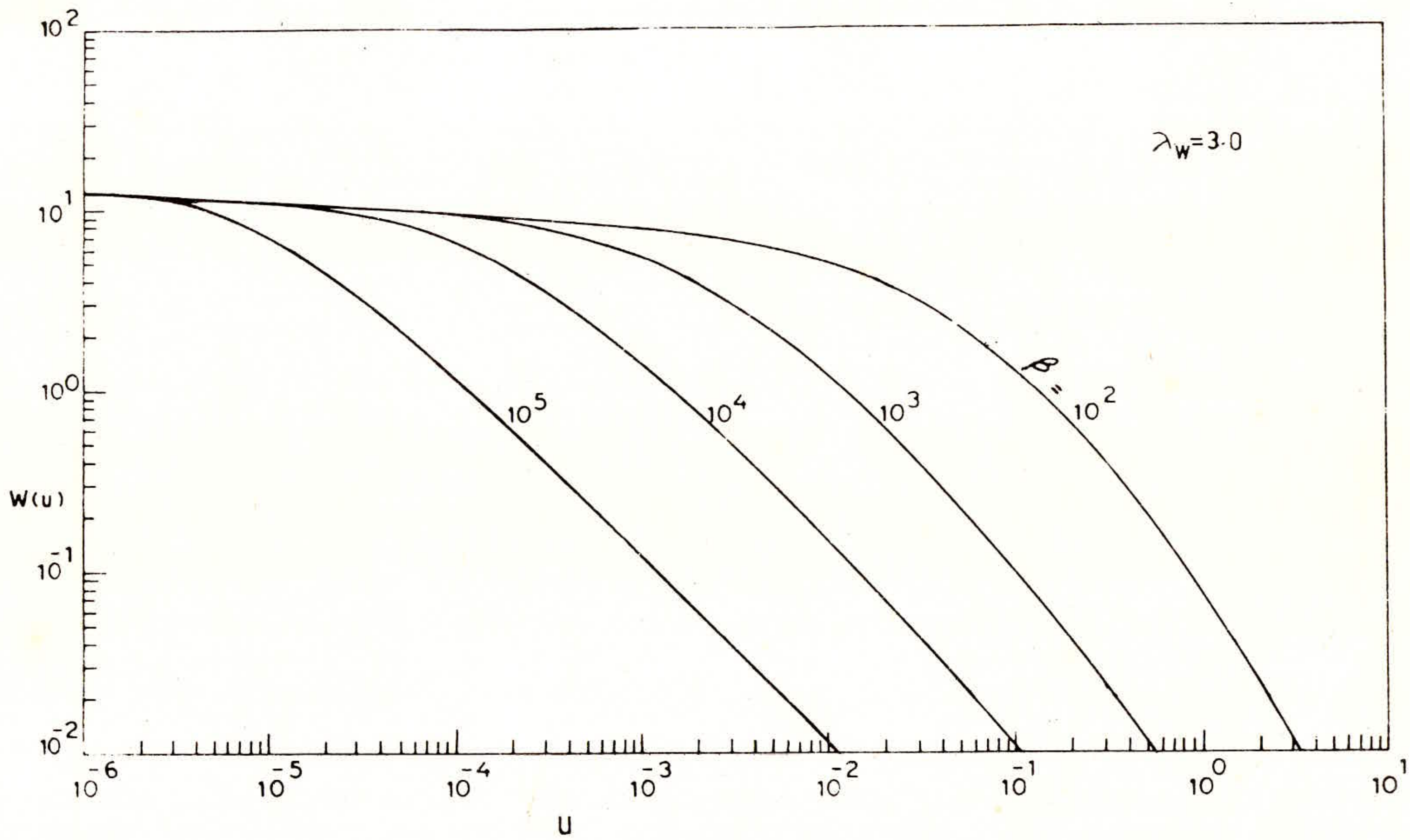

FIG. 2 TYPE CURVES FOR A FULLY SCREENED WELL WITH STORAGE IN A CONFINED AQUIFER- NON DARCY FLOW

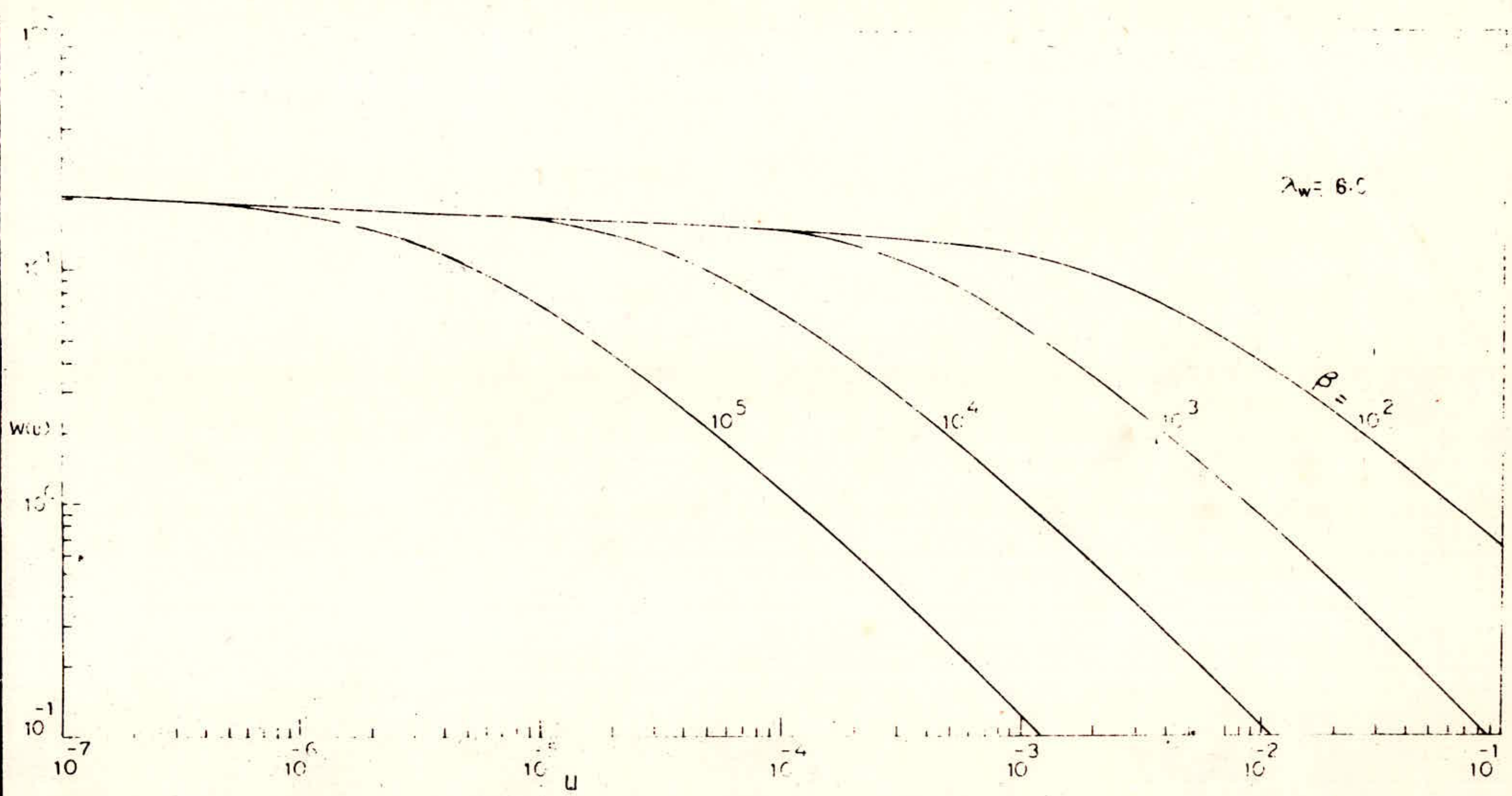

FIG. 3 TYPE CURVES FOR A FULLY SCREENED WELL WITH STORAGE IN A CONFINED AQUIFER – NON DARCY FLOW

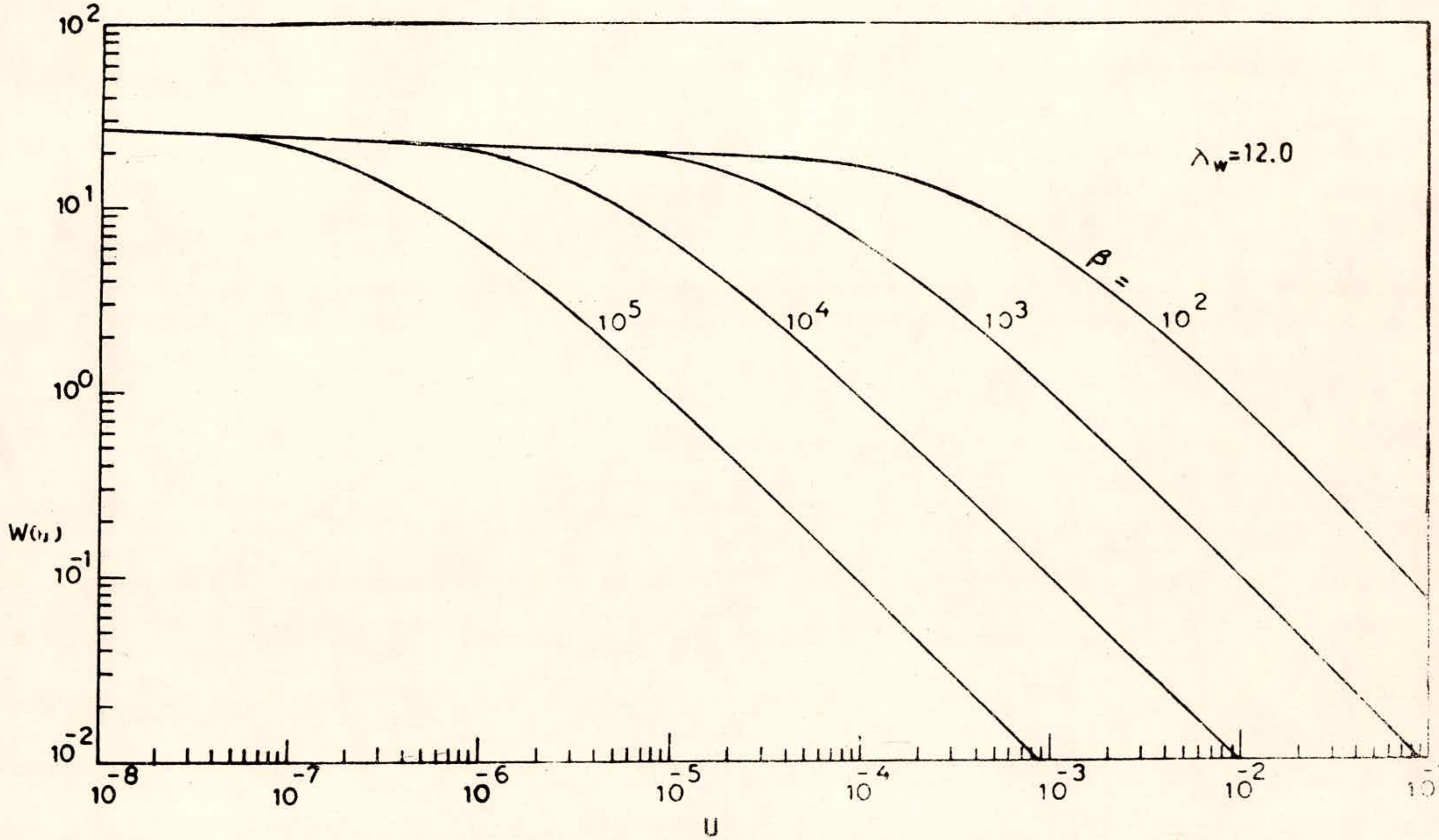

FIG. 4 TYPE CURVES FOR A FULLY SCREENED WELL WITH STORAGE IN A CONFINED AQUIFER - NON DARCY FLOW