International Conference "Water, Environment, Energy and Society" (WEES–2009)
New Delhi, 12–16 January 2009

Testing Weibull Distribution for Deriving Synthetic Unit Hydrograph Shapes with New Regionalization Techniques

P.K. Bhunya, S.K. Jain, P.K. Agarwal and D.S. Rathore
National Institute of Hydrology
Roorkee, Uttarakhand, INDIA

ABSTRACT: The two-parameter weibull distribution is presented as an alternate for developing synthetic hydrographs after a review of the existing applications like the popular gamma distribution method. The suitability of weibull distribution as Synthetic Unit Hydrographs (SUH) is explored for two cases using field data (i) for limited data conditions, and (ii) for no data condition, i.e. ungauged catchment. To estimate the parameters of the weibull distribution, an analytical procedure is proposed instead of existing numerical solutions. Taking the data of fifty-six catchments from India, equations for peak flow and time-to-peak are developed using two new methods: (i) non-linear optimization, and (ii) artificial neural network. Both these methods use a set of non-dimensional groups formed out of geomorphological variables as inputs. The workability of the proposed method is checked using a test catchment. The weibull distribution parameters are related to statistical properties of the hydrograph considering an equivalent triangular hydrograph, which eventually helps in analyzing the parameter sensitivity. The results show that the weibull distribution is flexible and more adaptable to derive SUH in ungauged catchments.

INTRODUCTION

Rainfall-runoff process is of immense interest in hydrology. However, many components of this process are difficult to observe routinely and unambiguously, and require costly measuring facilities. Due to economic and other constraints, such facilities are scanty, particularly in developing countries. Hence, a number of techniques have been developed to determine runoff when limited data are available. Among these, the Synthetic Unit Hydrographs (SUH) are of great significance to determine runoff volume with respect to time, especially from ungauged catchments. The qualifier *synthetic* here denotes that the ordinates of the Unit Hydrograph (UH) are obtained without using catchment's rainfall-runoff data. These yield a smooth and single valued shape corresponding to unit runoff volume, which is essential for UH derivation. Flood Estimation Handbook (1999) provides a good review of several methods of SUH derivation. Two distinct approaches are followed for development of a SUH. The first approach uses an empirical method e.g. McCarthy (1938) and Snyder's method (Snyder, 1938), which utilize empirical equations to estimate salient points of the hydrograph, such as peak flow (Q_p) (L^3T^{-1}), lag time (t_L) (T), time base (t_B) (T), and UH widths at $0.5Q_p$ and $0.75Q_p$. Functional relationships of catchment characteristics are used for this purpose. One such relationship was proposed by Bernard (1935) who accomplished the transformation of rainfall into runoff through medium of distribution graphs, which was assumed as a function of catchment characteristics. Similar expressions were later given by Edson (1951), Gray (1961), and Haan *et al.* (1984) among others. All these methods begin by obtaining the salient points of the UH, and a smooth curve is fitted through these points to obtain a SUH; thus a degree of subjectivity is involved in such manual fittings, as this require simultaneous adjustments for the area under the UH to represent unit runoff volume. In contrast, the *SCS* (1957) method utilizes the land use, soil type, hydrologic, and antecedent moisture condition of the catchment to estimate the peak discharge (q_p) and t_p, and than the SUH shape is determined from an average dimensionless q/q_p *versus* t/t_p hydrograph; thus avoiding any manual fitting.

An alternate approach to the SUH derivation makes use of the functional relationship between the important points on Instantaneous Unit Hydrograph (IUH) and catchment characteristics; thus providing a scientific basis for the hydrograph fitting. Clark (1945) was probably the first to propose the IUH theory, and the concept was later used by Nash (1958, 1959) and Dooge (1959) to develop a unit hydrograph. While Nash (1958, 1959) and Dooge (1959) derived the IUH as a two parameter gamma distribution from cascade of n linear reservoirs, Clark (1945) derived it by routing the unit inflow in the form of time-area diagram, which is prepared from the isochronal map through a single reservoir. Depending on data availability Croley (1980), Aron and White (1982) and Bhunya *et al.* (2003, 2004, 2007) applied the gamma and beta distribution to develop a SUH. Similarly, HEC-1 (1990, 1994) and Kull *et al.* (1998) have used the Clark's concept in developing SUH for ungauged catchments, by fitting the hydrograph through selected salient points like (t_p, q_p), (t_p, t_i) or (q_p, t_i), where t_i is the point of inflection after the peak. These procedures were an improvement over the earlier methods which used a pure empirical method to derive the synthetic unit hydrograph, further the subjectivity that existed in manual fitting of the UH in earlier methods was eliminated. Rodriguez-Iturbe and Valdes (1979) combined both of these methods and expressed the IUH in terms of geomorphological parameters as well as the transition state probability matrix of the water flow. The final expressions of the SUH were obtained by regression of the peak as well as time-to-peak of IUH derived from the analytic solutions for a wide range of parameters with that of the geomorphologic characteristics and flow velocities. Later, Gupta *et al.* (1980) modified this procedure for deriving an IUH, known as Geomorphological Instantaneous Unit Hydrograph (GIUH). This concept is frequently used in ungauged catchments. Rosso (1983) derived the Nash parameters in relation to the Horton order ratios using a power regression and developed the SUH. Similarly, Jueyi Sui (2005) and Kumar *et al.* (2005) have used the GIUH based Clark's model to develop flood hydrographs in ungauged catchments.

In SUH derivation, one of the important steps is the estimation of one or two key points on the UH (or IUH), through which the hydrograph is fitted. To achieve this objective, relationships are sought from the magnitude of the key parameters, such as q_p, t_p, t_B, t_i or t_L of the IUH, and selected catchment characteristics which can be measured from a topographic map, and generalized rainfall statistics. The preferred technique for developing such relationships has invariably been multiple linear regression analysis (see Hall *et al.*, 2001). This paper represents an extension of the earlier works in two ways: (i) uses a two-parameter Weibull distribution for fitting the UH and a new method is proposed for the estimation of distribution parameters; (ii) extends the multiple linear regression techniques to a non-linear optimization using non-dimensional groups of geomorphological variables for the regionalization of IUH parameters using data of fifty-six catchments; (iii) explores the potential of the artificial neural networks in constructing the regionalization of IUH parameters using the above data base; (iv) test the present method with existing two-parameter gamma distribution for SUH derivation. The workability of this approach in SUH derivation is demonstrated using data of an Indian catchment, not used for regionalization (or treating them as ungauged).

SYNTHETIC UNIT HYDROGRAPH METHODS

Two-Parameter Gamma Distribution

Using the concept of cascade of n-linear reservoirs, Nash (1959) and Dooge (1959) derived the gamma-form expression for IUH as,

$$q = \frac{1}{K\Gamma n}\left(\frac{t}{K}\right)^{n-1} e^{\frac{-t}{K}} \qquad \ldots (1)$$

where n and K are the number of reservoirs and the storage coefficient, respectively, and these describe the shape of IUH; and q is the depth of runoff per unit time per unit effective rainfall expressed in mm/hr/mm. Eqn. 1 is used for SUH derivation from known n and K. Defining q_p as the peak discharge and t_p as the time-to-peak in hr, a simple relation between n and β ($= q_p t_p$) of the gamma distribution for SUH derivation is given as follows (Bhunya *et al.*, 2003),

$$n = 5.53\,\beta^{1.75} + 1.04 \qquad 0.01 < \beta < 0.35$$

$$n = 6.29\,\beta^{1.998} + 1.157 \qquad \beta \geq 0.35 \qquad \ldots (2)$$

where and $\beta = q_p\, t_p$. Eqn. 2 can be used to estimate the shape of the SUH for known values of q_p and t_p.

Weibull Distribution

The pdf of a two-parameter Weibull distribution (Figure 1) is given by *Weibull* (1939) as,

$$f(t) = (a/b)(t/b)^{b-1} e^{-(t/b)^a};\ t > 0 \qquad \ldots (3)$$

Here, the scale parameter $a > 0$ and the shape parameter $b > 0$. Mean (μ) and variance (σ^2) of the pdf are given by,

$$\mu = b\Gamma(1+1/a);$$

$$\sigma^2 = b^2\Gamma(1+2/a) - b^2[\Gamma(1+1/a]^2 \qquad \ldots (4)$$

The cumulative distribution function (cdf) is given as,

$$F(t) = 1 - e^{-(t/b)^a} \qquad \ldots (5)$$

As $t \to \infty$, $F(t) = 1$. This condition meets the criterion for UH description (Sherman, 1941). Considering the UH similar to the Weibull distribution with discharge ordinate (q) on the y-axis and x-axis as time (t), (3) can be used to describe an IUH as,

$$h(t) = (a/b)(t/b)^{b-1} e^{-(t/b)^a} \quad \ldots (6)$$

For conditions at peak; q_p and t_p can be obtained as,

$$t_p = b[(a-1)/a]^{1/a} \text{ or } b = [a/(a-1)]^{1/a} t_p \quad \ldots (7)$$

$$q_p = \frac{a}{t_p[a/(a-1)]^{1/a}} \left(\frac{a-1}{a}\right)^{(a-1)/a} e^{-(a-1)/a} \quad \ldots (8)$$

Substituting $d = (a-1)/a$ in the above equation, and rearranging, $q_p t_p$ can be expressed as,

$$q_p t_p = \beta = 1/(1-d)(d^{1-d})(d^d)(e^{-d}) = (e^{-d} d)/(1-d) \quad \ldots (9)$$

Expanding the exponential in (9) up to second term, it can be simplified to the following form,

$$a^3 - (1+e\beta)a^2 + (e\beta)a - 1/2(e\beta) = 0 \quad \ldots (10)$$

This is a cubic equation in a, whose real roots gives the Weibull distribution parameters as (Abramowitz and Stegun, 1964),

$$a = (u+v) - A/3 \quad \ldots (11)$$

where,

$$u = [r + (s^3 + r^2)^{1/2}]^{1/3};$$
$$v = [r - (s^3 + r^2)^{1/2}]^{1/3}; s = B/3 - A^2/9;$$
$$r = (AB - 3C)/6 - A^3/27$$

and,

$$A = -(1+e\beta);\ B = e\beta;\ \text{and } C = -e\beta/2. \quad \ldots (12)$$

An alternate solution of (9) can be obtained using a numerical procedure as follows:

1. Generate d using a random number scheme within a practical limit,
2. Estimate corresponding β-value from (9), and
3. Fit an equation to the ordered pair (d, β).

For this scheme d is simulated using a uniform distribution random number generator scheme available on any standard spreadsheet, corresponding to the limit of $q_p t_p$ (or β) between 0.01 and 2. It is noted here that β-values less than 0.01 are seldom experienced in field and the maximum value is rarely found to be greater than one (see Singh, 2000). For $d \leq 0.67$ (corresponding to $\beta \leq 1.104$), the (d, β) sets perfectly fit a third order polynomial of the following form,

$$d = 0.0039(\beta^3) - 0.4427(\beta^2) + 1.099(\beta) - 0.0048$$
$$\text{for } \beta \leq 1.104,\ d \leq 0.67 \quad \ldots (13)$$

for $d > 0.67$ (corresponding to $\beta > 1.104$) the scatter plot of (d, β) is observed to follow a inverse power function, therefore, the value of d is assumed as a function of β given as,

$$d = \frac{m_1}{(m_2 + m_3\beta^{m_4})} \quad \ldots (14)$$

where m_i are the parameters of (14) which is obtained using a suitable optimization scheme. For this the Marquardt algorithm (Marquardt, 1962) was employed for parameter estimation and the form of (14) is obtained as follows,

$$d = 1.1805/[1.192 + 0.591\beta^{-1.241}],$$
$$\beta > 1.104,\ d > 0.67 \quad \ldots (15)$$

Thus for a given value of G* or β (estimated from the geomorphologic parameters of the catchment), the shape parameter 'a' of Weibull distribution can be estimated from (11) and (15) by analytical and numerical procedures, respectively. Now the shape parameter estimate of the Weibull distribution is substituted in (7) along with t_p estimated from (6) to get the expression for scale parameter, b, as follows,

$$b_* = 1.584[a/(a-1)]^{1/a}(R_B/R_A)^{0.55} R_L^{-0.38} v^{-1} L \quad \ldots (16)$$

where $b_* = bv^{-1}L$ is non dimensional. The correctness of the proposed equations for 2PWD parameter estimates are checked in the following section.

REGIONALIZATION OF UH PARAMETERS

So far the discussion was focused on deriving the shape of the SUH; when two or three salient points on the UH (or IUH) are available, e.g., time-to-peak, peak flow or base of the hydrograph. Through these points, the hydrograph is fitted to give it a complete shape. Therefore, the point of concern that remains is to get these salient points of the UH for ungauged catchments. Mostly, such relationships are developed using multiple linear regression analysis (Hall *et al.*, 2001), where a functional relationship is derived between the important unit hydrograph parameters and catchment characteristics. A similar approach was applied to regional analysis of floods by Tasker *et al.* (1996) and Swamee *et al.* (1995) to estimate flood quintiles, and mean flood estimation in ungauged catchments, respectively. The present study attempts to relate q_p and t_p with non-dimensional groups of geomorphological parameters using (i) a non-linear regression method, and (ii) Artificial Neural Network (ANN) techniques using the data of fifty-six catchments.

Study Area

In this study, the following catchments from India were considered for application: *Lower Narmada* and *Tapi* sub-zone (3B): Geographic location of sub-zone extends from 73° 14' to 76° 30' E longitude and 20° 19' to 23° 0' N latitude. This is a sub-zone delineated by CWC (1982a) and comprises of a number of small bridge catchments with area varying from 53 to 828 km^2.

Mahanadi Sub-zone (3D)

Geographic location of sub-zone extends from 80° 25' to 87° E longitude and 19° 15' to 23° 35' N latitude. This is a sub-zone delineated by CWC (1982b) and comprises of *Mahanadi*, *Baitarani* and *Brahmani* among its major catchments. The study considers short-term rainfall runoff data of small bridge catchments with area varying from 19 to 1150 km^2.

Lower Godavari Catchments (Sub-zone 3 F)

Sub-zone is bounded by *Upper Narmada* catchment, *Tapi* catchment, *Krishna*, and *Coromondel* sub-zones. The total area of the sub-zone is about 174,000 sq. km. The sub-zone is L-shaped and extends from 76° 83' E longitude and 17° 23' N latitude. The sub-zones 3B and 3F are further sub-divided for hydro meteorological studies (CWC, 1980; CWC, 1982) and the available data of total 22 watersheds of these sub-zones were used in this study.

Myntdu-Leska River Basin

Basin is located in Jaintia hills District of Meghalaya, in the northeastern part of India, in the southern slope of the state adjoining Bangladesh. Its geographic location extends from 92° 15' to 92° 30' E longitude and 25° 10' to 25° 17' N latitude. The area is narrow and steep, lying between central upland falls of the hills of Meghalaya. The catchment area is about 350 km^2 and elevation range varies from about 1372 m to 595 m. Ths study area has been shown in Figure 1.

Bridge Catchment No. 253

The bridge catchment on Tyria stream of Narmada river at Gondia - Jabalpur railway line lies approximately near 80° E longitude and 23° 5' N. The catchment area is equal to 114.22 km^2, and the stream length up to the bridge is 35.42 km. These data were used by Lohani *et al.* (2001).

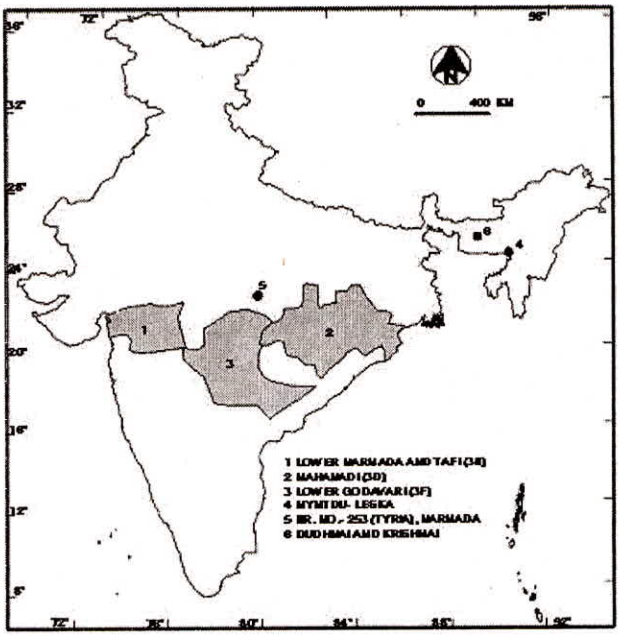

Fig. 1: Study area

The data statistics is given in Table 1. Since the results of the present study are compared with other popular synthetic methods, the catchments are selected to lie in the small to medium range (100–500 km^2); 33 catchments are less than 250 km^2. It may be noted here that zone 3b, d, and f are homogeneous regions in respect to their flood producing mechanism (CWC, 1980; 1982), however, the data sets as a whole are not subjected to any separate homogeneity tests; therefore, the study area is non-homogeneous.

Non-Linear Optimization Method

Estimation of q_p

This section focuses on development of an empirical model for q_p estimation for the study area, which can be used for any ungauged catchment within the study area. For this, the following non-dimensional groups are considered: (i) A/L^2, (ii) L/L_c, and (iii) $(L/(L_cS^{1/2}))$. The variable S here denotes mean catchment slope, and is estimated as ratio of difference in elevations at 0.85 and 0.10 times basin length from mouth of the basin to 0.75 times the basin length (Olivera, 2002). Elevations are measured along the line of maximum length of the catchment. Mean catchment slope can also be calculated manually or within a GIS as average of slope at many random or regular points in the catchment.

Table 1: Summary of the Catchment and Unit Hydrograph Characteristics of the Study Area

Catchment /Zone	*No. of Catchments*		*Range of*					
	Total	*<250 km²*	*A (km²)*	*S (km/m)*	*L (km.)*	*L_c (km)*	*Q_p (m³/s)*	*t_p (hr.)*
3B sub zone	17	9	53–828	1.3–8.4	18.7–74.5	7.2–37	50–300	1.5–6.5
3D sub zone	16	10	47–3108	0.6–9	12.5–96.6	7–52	17.4–560	4.5–16.5
3F sub zone	22	13	4–969	1.3–9.1	10–92.3	5–43	41–650	1.2–11.5
Myntdu-Leska	1	–	350	4.2	51.8	27.8	118.3	5
Br. No. 253	1	1	114.2	2.5	35.4	16.9	54.6	5

In formulating these non-dimensional groups, two basic issues are taken care of: (i) the physical significance of non-dimensional groups, and (ii) the improvement of model accuracy compared to existing empirical relationships (or regional formulae). Regarding the first issue, the non-dimensional groups (A/L^2) is the *form factor* used to quantify the degree of similarity of drainage catchment shapes (in a region), which influence the flood hydrograph shape (Olivera, 2002). Similarly, L/L_c gives an idea about the *eccentricity* of the catchment (Black, 1991), and the mean slope is related to the velocity of flow and attenuation time of the hydrograph. Earlier studies of Gupta *et al.* (1980), CWC (1980; 1982a, b), and Hall *et al.* (2001) have used geomorphologic variables to form non-dimensional groups e.g. (L/L_c) and ($L/(L_cS^{1/2})$) to develop regional formula for q_p estimation. The second issue addresses the number of geomorphological parameters that need to be considered for developing the model. Since the flood peak or time-to-peak or any other UH parameter not only depends on one or two physical characteristics of the catchment, but also on many other factors, the model need to assess the optimum number of catchment parameters that need to be considered to yield precise results. While considering the number and nature of catchment parameters, it is to be ensured that they can be computed easily from a toposheets or any map by a user, who ought to use this formula for an ungauged catchment. It is the second issue that has gained precedence in hydrological studies, including the study on mean annual flood estimation by Wong (1979) and Swamee (1990).

To obtain the empirical model, it is necessary to examine the possible relationships among the non-dimensional groups; therefore, the correlation test is supplemented by the paired *t*-test of mean of correlation to infer the significance of the above variables, following the null hypothesis (Montgomery and Runger, 1994). The results (Table 2) shows the correlation to be low (0.45 and 0.5) except $L/L_C - L/(L_C S^{1/2})$ with a moderately high correlation of 0.67, which is however rejected with paired t-test. Therefore, it is appropriate to consider all the three groups for developing the model. This is further proved using the Buckingham π-theorem (Langhaar, 1951) with mass (M), length (L) and time (T) as the repetitive variables, which forms the following empirical model for q_p estimation,

$$[q_p] = a_0\left(\frac{1}{T}\right)\left(\frac{A}{L^2}\right)^{a_1}\left(\frac{L}{L_c}\right)^{a_2}\left(\frac{L}{L_C S^{1/2}}\right)^{a_3} \quad \text{... (17)}$$

Considering the aspects discussed above, the following empirical model is proposed for the estimation of q_p,

$$[q_p]_i = a_0\left(\frac{1}{t_p}\right)_i\left(\frac{A}{L^2}\right)_i^{a_1}\left(\frac{L}{L_c}\right)_i^{a_2}\left(\frac{L}{L_C S^{1/2}}\right)_i^{a_3} \quad \text{... (18)}$$

where suffix *i* stands for the i^{th} data set, and a_0–a_4 are the fitted coefficients. Time-to-peak, t_p is substituted for dimension T because it is known to influence the peak discharge in a significant way (Nash, 1959; Dooge, 1959; Gray, 1961) and Rodriguez-Iturbe and Valdes, 1979). It may be highlighted that the proposed

Table 2: Significance-Test for Non-Dimensional Groups for the Whole Study Area

Sl. No. (1)	*Paired Non-Dimensional Groups* (2)	*Correlation* (3)	*Paired t-Statistics (t_o)* (4)	*t_{cr}* (5)	*Accepted/Rejected Using t-test* (6)
1.	$L./L_C - A/L^2$	0.45	5.302	1.99	Rejected
2.	$A/.L^2 - L/(L_C\,S^{1/2})$	0.5	6.015	1.99	Rejected
3.	$L/L_C - L/(L_C\,S^{1/2})$	0.67	8.52	1.99	Rejected

model (Eqn. 18) satisfactorily addresses some key issues: (i) it is not unit dependent, (ii) more catchment parameters are considered, (iii) the non-dimensional groups have a physical significance, and (iv) the catchment parameters considered here can easily be interpreted from toposheets and imageries.

Taking the observed data sets (Table 1), the Marquardt algorithm of non-linear optimization (Marquardt, 1962) was employed for estimation of parameters of Eqn. 18. For optimization, the objective was to minimize the sum of square of errors (E) between the observed and computed values of q_p. The goodness-of-fit is evaluated using the coefficient of determination (COD or R^2) (Mendenhall and Sincich, 1988) given as,

$$COD(R^2) = 1 - \frac{\sum_{i=1}^{N}(y_o - y_c)^2}{\sum_{i=1}^{N}(y_o - y_m)^2} \quad \ldots (19)$$

where y is the variable under consideration; N is the number of observations; and the subscripts o, c, and m refer to the observed, computed, and mean values, respectively. The resultant parameters of Eqn.18 are given in Table 6a for two cases: (i) for catchment area less than 250 km^2, and (ii) for area greater than 250 km^2; this categorization on the basis of catchment areas yields the maximum COD.

The results of R^2 for the two cases are 0.88 and 0.85, respectively, which indicates that the calibration is good. The validation of Eqn. 18 is done using data of two catchments as reported in Table 6b, which shows error less than 3 percent.

Estimation of t_p

Many empirical relations for computation of time-to-peak have been proposed in the past e.g. *Snyder* (1938) derived time-to-peak as a function of L, L_c, and a regional non-dimensional coefficient, which is related to t_p (SCS, 1957). Similarly, Linsley *et al.* (1975) and Flood Studies Report (1975) expressed t_p as a function of L, L_c and S, and Rodriguez-Iturbe and Valdes (1979) have related it to Horton order ratios (Horton, 1945) and L. For the present case, a multiple regression analysis was initially used to check the correlation between time-to-peak and each of the non-dimensional groups. The results did not show any strong correlation between the two; therefore the regression was tried by employing the geomorphological variables individually, which did not improve the results either. This motivated the authors to try a new variable, i.e. time of concentration (t_c) to relate t_p. The time of concentration is defined as the time taken by a water particle to travel from farthest point of the catchment to the outlet (Chow, 1965) and t_c behaves in a similar way as the time-to-peak for a UH. Most widely used relationship for computing t_c proposed for use with ungauged watersheds was developed by Ramser (1927) (quoted in Haan *et al.*, 1984) as follows,

$$t_c = 0.02 L_c^{0.77} S^{-0.385} \quad \ldots (20)$$

where t_c is the time of concentration in minutes, S denotes the average slope of channel in m/m. A simple scatter plot between time of concentration and time-to-peak of the data from the study area is reported in Figure 2 that shows a R^2 equal to 0.65 using a linear fit. Thus, for the present case, the proposed empirical model is as follows,

$$t_p = f(t_c, c_i) \quad \ldots (21)$$

where f is any arbitrary function and c_i are the parameters that need to be calibrated for the study area. Using the data for forty-one catchments, the final form of Eqn. 22 is derived as follows,

$$t_p = 193.786\left(\frac{t_c}{60}\right)^{1.0486} \quad \ldots (22)$$

The results of validation using data of fifteen catchments give a $R^2 = 0.88$. Eqns. 18 and 22 can be used to express parameter β as a function of only geomorphological variables. The same data as used in case of validation of time-to-peak are used for validation of β, and the results shows a fit with $R^2 = 0.68$. The following section discusses the empirical model development of q_p and t_p estimations using ANN technique.

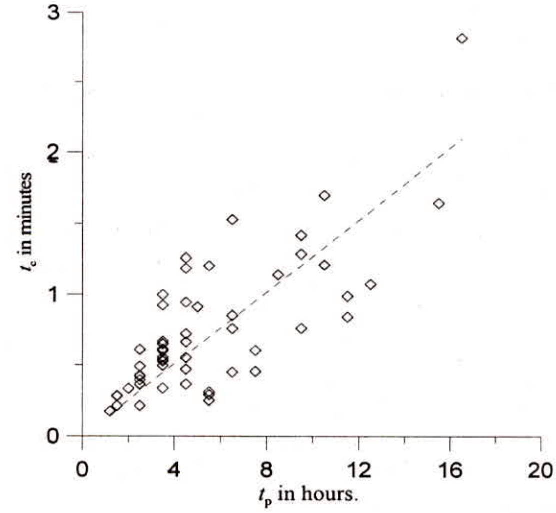

Fig. 2: Correlation between t_p and t_c for the study area

Artificial Neural Network (ANN) Method

In view of the versatility of artificial neural networks in modeling non-linear systems, this technique was also employed to determine q_p and t_p. Since the theory of ANN has been extensively discussed in the contemporary literature, the same is not repeated here. Detailed review of theory and application of ANN was provided by ASCE (2000a, b) and Maier and Dandy (2000). Two feed-forward three-layer ANNs were developed in this study: a) to determine t_p using the data of t_c, and b) to determine q_p using the data of t_p and a non-dimensional factor $A/(L_c^2\ S^{0.5})$.

The first ANN had a single input and a single output. Total 55 pairs of data were available. In the data, t_c ranged from 1.2 hours to 12.5 hours and t_p ranged from 0.17 to 2.8 hours. The data were standardized by the following formula,

$$N_i = \frac{R_i}{(Max_i * a) - (Min_i * b)} \quad \ldots (23)$$

where R_i is the i^{th} input data; N_i is the corresponding standardized value; Min_i is the minimum of all the values of the variable R; Max_i is the maximum of all the values of the variable R; and a and b are constants which control the range of standardization. In the present case, $a = 1.2$ and $b = 0.8$.

During training, the number of neurons in the hidden layer was systematically changed and the best results were obtained using 5 neurons. Levenberg-Marquardt method was used for training. The first 40 data were used for training and the last 15 for testing. For the training data set, RMSE was 0.0904 and the model efficiency was 0.5229 while for the test data set, RMSE was 0.2478 and the model efficiency was 0.1851. It was noticed that in the data, values in the higher range mostly occurred in the later part of the set. Hence, another ANN was trained in which the data for the last 40 sets was used for training and that for the first 15 for testing. In this case, the training set RMSE was 0.1075 and model efficiency 0.6983 but for the test data, RMSE was 0.1056 and model efficiency –0.2175. This indicates that the ANN is not able to properly learn the underlying behavior of the data, probably due to insufficient inputs.

The second ANN had two inputs: t_p and $A/(L_c^2\ S^{0.5})$ and a single output. Data of 56 catchments were available. Here, the first 40 sets were used for training and the last 16 for testing. For the training data, RMSE was 0.0531 and model efficiency 0.942 but for the test data, RMSE was 0.1129 and model efficiency 0.0025, indicating a poor training. When the data of the last 40 patterns was used for training and that of the first 16 for test, the results were still not good. To investigate the reason behind this, a scatter plot of all the data was prepared and it was noticed that two data points had an altogether different behavior than the remaining. Ignoring these two patterns, 54 sets remained out of which, 40 were used for training and 14 for test. When the ANN was trained using this new data set, RMSE for the training set was 0.0551 and efficiency 0.9155 while for the test data, RMSE was 0.0936 and efficiency 0.8262. This shows that now the ANN was able to correctly learn the underlying behavior of the data.

When q_p for the Bridge catchment No. 253 (Table 4) was estimated by the use of this ANN, it turned out to be 0.179 and t_p was estimated to be 5 hrs, which is very close to the observed values given in Table 4.

APPLICATION

Utilizing the data of Bridge catchment No. 253, the applicability of the proposed method in deriving SUH is examined for two cases: (i) Case A investigates the workability of the proposed method when it has partial data, i.e. few observed data from the observed hydrograph e.g. Q_p and t_p are used for the analysis, and (ii) Case B derives SUH considering the catchment as an ungauged catchment, i.e. with no data. Two selected flood events are considered for this case.

Case A: The rainfall-runoff flood hydrograph that occurred over the Bridge catchment No. 253 on 23rd Aug, 1996 was considered for this case. The salient points of the flood hydrograph were as follows: Q_p = 82.5 m^3/s, t_p = 5 hrs, t_B = 21 hrs. The unit hydrograph for the catchment was taken from the study of Lohani and Singh (2001). The parameters of the beta distribution were estimated using Eqns. 9 and 13, and are shown in Table 2 i.e. t_B = 105 hrs, α = 21, p = 96.17, r = 5.51. The observed and the derived flood hydrograph event for this catchment are shown in Figure 3(a). To check the performance of this method, the goodness-of-fit is further evaluated using the ratio (STDER) of the absolute sum of non-matching areas to the total hydrograph area, expressed mathematically as (US Army Crops of engineers, 1990),

$$STDER = \left[\frac{\sum_{i=1}^{N}(q_{oi} - q_{ci})^2 w_i}{N}\right]^{\frac{1}{2}} ; \ w_i = \frac{(q_{oi} + q_{av})}{2q_{av}} \quad \ldots (24)$$

Table 3: Example for Estimating Weibull Parameters Eqns. (9) and (13)

q_p (Hr^{-1})	t_p (Hr.)	$\beta = q_p\, t_p$	Assumed t_B in hrs.	$\alpha = t_B/t_p$	r Assumed	p using Eqn. (9)	Computed β (Eqn. 13)
0.1727	5	0.8637	100	20	4	65	0.7017
			100	20	5	82	0.8141
			105	21	5.2	90.2	0.833
			105	21	5.51	96.17	0.8637

where q_i = i^{th} ordinate of the observed hydrograph, q_{ci} = i^{th} ordinate of the computed hydrograph, w_i = weighted value of i^{th} hydrograph ordinates, q_{av}= average of the observed hydrograph ordinates, and N = total number of hydrograph ordinates. Since the computed w_i-values (Eqn. 24) are larger for higher q-values, the resulting high STDER value signifies larger non-matching areas on the upper portion of the hydrograph to non-matching areas in the lower portion below q_v. A low value of STDER-value represents a good-fit, and vice versa; STDER equal to zero represents a perfect fit. For the Bridge catchment, STDER = 9.7, indicating a good fit.

Taking the same values of q_p and t_p that was used above, the SUH was derived using the gamma, Gray, and SCS methods. The gamma distribution parameters were computed using Eqn. 2 as follows: n = 5.81 and K = 1.04 and the computed flood hydrograph for the bridge catchment is derived using Eqn. 1, and is reported in Figure 3. Similarly, the STDER for the gamma, SCS and the Gray's methods were found to be 14.1, 5.4 and 13.6, respectively. This means that the flood hydrograph derived by using the SCS method is closer to the observed values followed by beta, Gray and gamma methods. To further check the efficiency of these methods, the catchment is considered ungauged in the following case.

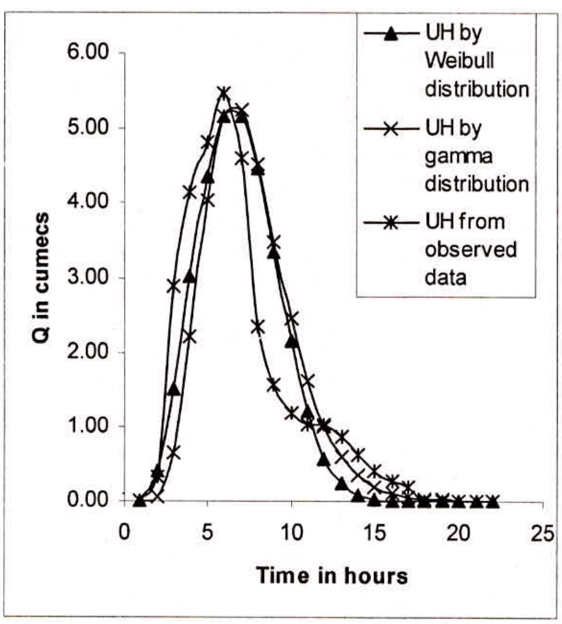

Fig. 3(a): Comparison of UHs using Weibull and Gamma pdfs

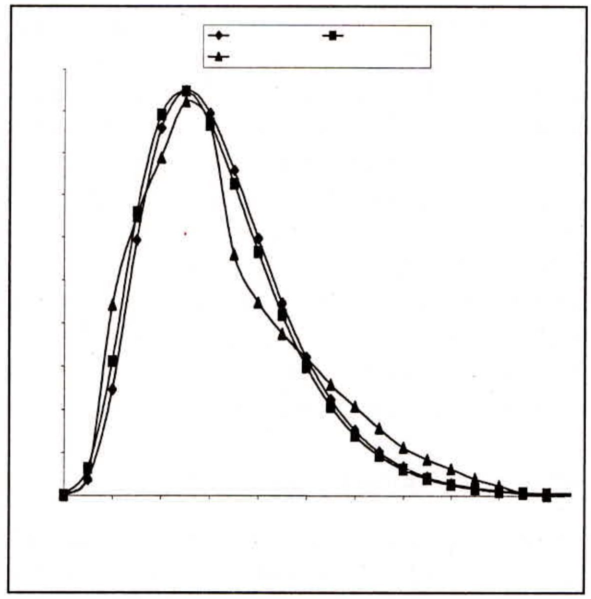

Fig. 3(b): Comparison of UHs using Weibull and Gamma pdfs considering the bridge catchment as ungauged

Case B: Considering the Bridge catchment as ungauged, i.e., for no data situation, q_p and t_p of the SUH are estimated using the empirical relationships derived for the study area. For the present case, catchment variables are: L = 35.42 km, L_c = 19.6 km, S = 3.7 m/km. Using these above data in Eqn. 22, time-to-peak is computed as 4.6 hrs, and the corresponding q_p (Eqn. 16) is equal to 0.176 hr^{-1}. The parameters of the SUH (Eqn. 6) were computed using Eqn. 13 as follows: p = 89.31 and r = 5.016 and the corresponding flood hydrograph is shown in Figure 3(b), which shows a close match with the observed UH. The figure also shows the SUH derived by SCS method where the q_p and t_p values are calculated using Eqn. 3 and the shape of SUH from non-dimensional q/q_p versus t/t_p hydrograph (SCS, 1957). The SCS method overestimates the peak flow and time-to-peak values; the resulting STDER values for beta based regression approach and SCS methods are 11.1 and 32.7, respectively. It can be seen that the SUH with SCS method is deviating from the observed UH both in rising and falling limb.

Using the ANN approach the two points of the UH are calculated here for no data situation i.e. considering the catchment as ungauged. Taking the catchment variables the non- dimensional groups are computed, and are used in the ANN method to derive t_p as 5 hrs, and q_p equal to 0.179 hr^{-1}. For this the parameters of the beta distribution are calculated as follows: $p = 98.3$ and $r = 5.815$ and the corresponding flood hydrograph is shown in Figure 3(b). The STDER due to beta based ANN method is found to be 14.4 i.e. slightly higher than that obtained with the regression approach. Lastly Snyder's method is employed to derive the flood hydrograph using Eqn. 1, and the corresponding STDER is 18.6. All the computed flood hydrographs are shown in Figure 3(b).

DISCUSSION OF RESULTS

The analysis presented here focused on the use of beta distribution for deriving synthetic unit hydrographs for ungauged catchments. To compute the parameters of the beta distribution, an approximate analytical solution is derived that works accurately when compared to the existing numerical solutions of Bhunya *et al.* (2004) using field data. To examine the possible advantage of using beta distribution in developing unit hydrograph, the results of the flood hydrographs are compared with the existing methods such as SCS (1957), gamma distribution and Gray's method using a suitable goodness of fit criteria. The second section of this study attempts to regionalize the unit hydrograph parameters viz. peak flow of the UH and time-to-peak using non-dimensional variables comprising of geomorphological chrematistics of the catchments. The regional formulae for estimation of peak flow and time-to-peak of an UH for ungauged catchments was derived using these non-dimensional variables by applying both non-linear regression approach and ANN technique. For this data of fifty-six catchments were used, and the performance was checked using data of two test catchments. The results showed high values of R^2 in the vicinity of 0.8. The following section analyzes the sensitivity of the parameters to peak flow computation, and further makes an analogy by a approximating the unit hydrograph to a triangle.

Sensitivity of Beta Distribution Parameters

Since q_p is a function of p, r, t_B, and t_p (Eqn. 13) the sensitivity of the parameters to q_p estimates can be evaluated using the partial derivative as follows,

$$\frac{\partial q_p}{\partial t_p} = \frac{\partial \lfloor f(r, p, t_B, t_p) \rfloor}{\partial t_p} \quad \ldots (25)$$

where f is any arbitrary function. Eqn. 25 is evaluated numerically using a simulation procedure for two different cases: (i) keeping p and t_B constant, and varying r for different values of t_p, (ii) keeping r and t_B constant, and varying p for different values of t_p. It is observed from the results (Figure 4(a)) that for small values of p, the curve is flat i.e. rate of decrease in q_p is small, compared to higher p-values where the curve is steep with higher $\partial q_p / \partial t_p$. Further, the q_p–t_p curve follows an exponential relationship of the following form,

$$q_p = a_p (t_p)^{b_p} \quad \ldots (26a)$$

where a_p and b_p are the parameters that vary with p and are analysed for p in the range of 10 to 90. Similarly, q_p–t_p curve fits an exponential curve given by,

$$q_p = a_r (t_p)^{b_r} \quad \ldots (26b)$$

where a_r and b_r are the parameters that vary with r and are given in Table 7 for r in the range of 1.5 to 5. An inspection of the two graphs shows that q_p is more sensitive to r than p; for 25 percent increase in r value, the corresponding increase in q_p is 33 percent whereas for the same increase in p the increase in q_p is 13 percent. The relationship between q_p and t_p (Eqn. 26) is analogous to SUH derived by two parameter gamma distribution (Nash, 1959; Dooge, 1959) in the following way. When $t = t_p$, $q = q_p$; the beta pdf equation can be simplified to the following form,

$$q_p = \frac{(n-1)^{(n-1)} e^{-(n-1)}}{\Gamma(n-1)} t_p^{-1} = a_n t_p^{b_n}; b_n = -1 \quad \ldots (27a)$$

where,

$$a_n = \frac{(n-1)^{(n-1)} e^{-(n-1)}}{\Gamma(n-1)} \quad \ldots (27b)$$

Comparing Eqns. (26a) and (27), it can be observed that parameters a_r and a_n, b_r and b_n are similar e.g. for $r = 5$, $a_p = 0.7824$, and $b_p = -0.97$; for $n = 5.01$, $a_n = 0.7824$ and $b_n = -1$. This shows that the parameter r in beta distribution is similar to the parameter n in gamma distribution, however, b_r in beta distribution has a variation though small in magnitude r as observed from the results in contrast to b_n which is a constant (= –1). This might be one of the factors that contribute to the flexibility in the shape SUH while using the beta distribution. This hypothesis is further proved in the following section while analyzing the sensitivity of these parameters using an analogous triangular UH.

Though the two parameter gamma *pdfs* also describe the UH shape well, the major limitation is its inability to yield a fixed t_B value which approaches infinity when q approaches zero. It means that the recession limb of UH (or *pdf*) nears right side asymptotically, approaches the x-axis rather than culminating at a finite t_B as required for UH. Therefore, while using a gamma *pdf* for SUH derivation, computations are carried out up to the point, considered as t_B, after which q is negligibly small; this observation is particularly valid for positively skewed gamma. Since beta distribution can skew on either side (positive and negative) similar to an UH encountered in practice, they are more flexible in description of SUH shape. This is also confirmed in applications of these methods to field data by giving least fitting errors.

SUH as an Analogous Triangular Hydrograph

In this section, the mean and variance of the unit hydrograph computed using an analogical triangular hydrograph are expressed in terms of time base and time-to-peak of the unit hydrograph. The approximate expression obtained by this procedure is then used to check the sensitivity of the beta distribution parameters.

Assuming a triangular UH, and its peak discharge, time-to-peak, and time base as parameters, the mean and variance of the UH is given as follows,

$$\text{Mean}\,(\mu) = \frac{\sum_{i=1}^{N}(q_i)(t_i)\Delta t}{\sum_{i=1}^{N} q_i \Delta t} = \left(t_{\text{p}} + t_{\text{B}}\right)/3 \qquad \dots (28a)$$

and

$$\text{Variance}\ (\sigma^2) = \frac{\sum_{i=1}^{N}(t-\mu)^2 (q_i)\Delta t}{\sum_{i=1}^{N} q_i \Delta t} \qquad \dots (28b)$$

where 'N' is the number of intervals of the time ordinate, 'Δt' is the time increment, t_i is the time up to the i^{th} interval, and q_i is the discharge at the i^{th} interval, derivable for any point of time t in the triangular UH as,

$$q_i = \frac{q_{\text{p}}}{t_{\text{p}}} t \quad 0 < t < t_p$$

$$q_i = q_{\text{p}} - \frac{(t_{\text{B}} - t)}{(t_{\text{B}} - t_{\text{p}})} q_{\text{p}} \quad t_p < t < t_B \qquad \dots (29)$$

In the limit $\Delta t \to 0$, the summation (Σ) in Eqn. 28 approximates the integral form as follows,

$$\mu = \int_0^{t_B} (q_t)(t)dt/(1/2q_p t_p)$$

$$\sigma^2 = \int_0^{t_B} (t-\mu)^2 (q_t)dt/(1/2q_p t_p) \qquad \dots (30)$$

where $q_t = q_i$. The substitution of Eqn. 29 in Eqn. 30 leads to,

$$\sigma^2 = C_\alpha (t_p)^2$$

$$C_\alpha = \frac{(\alpha + \alpha^2 - \alpha^3)}{30(1-\alpha)} \qquad \dots (31)$$

For the bridge catchment, μ and σ^2 calculated using the actual UH data are 7.76 and 8.76 hrs, and using the triangular approach equations (Eqns. 29a and 30) the moments of the UH are calculated as 7.33 and 8.45 hr^2, respectively. Therefore, the triangular analogy of the UH yields the moments of the hydrograph to a fair degree of accuracy. It can be verified that the μ–t_p relationship indicates t_p to behave like a location parameter. Similarly, variance gives the spread or dispersion of discharge ordinates of hydrograph. A high variation indicates that the discharge ordinates are evenly spread and a low variance means high discharge values cluster at the centroid of UH giving rise to a steep middle part. This is also evident from Eqn. 31; a large t_B and small t_p gives a large value of α and, in turn, high variance. It can occur if rising or recession limb varies asymptotically, showing high spread of 'q_i' ordinates. It can be checked that σ^2 increases more rapidly for any increase in t_p compared to the rate of increase with rise in α i.e. the dispersion of UH ordinates is largely sensitive to t_p than t_B. This is also evident from the $\sigma^2 - t_p$ relationship. Yue *et al.* (2002) have expressed the parameter c of the beta distribution (Eqn. 8), which can be computed using the method of moments as follows,

$$c = \left[\frac{\mu^2(1-\mu)}{\sigma^2} - \mu\right]\left(\frac{1}{\mu} - 1\right) \quad \ldots (32)$$

Replacing x as $1/t_p$ (since $\alpha = t_B/t_p$ and $t_B = 1$) the parameter r can be expressed as a function of μ, σ^2, t_p and t_B, however, this shall also give an approximate parameter estimate.

CONCLUSIONS

The following conclusions are drawn from the present study:

1. The beta distribution describes the SUH better than the existing synthetic methods for limited data and no data situations.
2. The parameter estimation for the beta distribution using the proposed analytical equation compares well with the similar results obtained earlier. The proposed method has an advantage over the existing numerical methods in that it allows study of relationship between the distribution function parameters and the UH parameters.
3. The non-linear optimization and ANN technique proposed for deriving the empirical regional equations showed good performance when used for ungauged catchments, even when the equations were derived using data from a non-homogeneous region.
4. Parameter r in beta distribution is similar to the parameter n in gamma distribution
5. The beta distribution parameters are related to q_p exponentially; for small values of p the rate of decrease in peak flow is small compared to higher p-values, further q_p is more sensitive to r than p.
6. Representing UH by an equivalent, the analysis shows t_p to behave like a location parameter. Further, the dispersion of UH ordinates were observed to be more sensitive to t_p than t_B.

REFERENCES

ASCE (2000a). 'Artificial neural networks in hydrology-I: Preliminary concepts', J. of Hydrologic Engg., ASCE, 5(2), 115–123.

ASCE (2000b). 'Artificial neural networks in hydrology-II: Hydrologic applications', J.of Hydrologic Engg., ASCE, 5(2), 124–137.

Aron, G. and White, E.L. (February, 1982). 'Fitting a gamma distribution over a synthetic unit hydrograph', Water Resources Bull., American Water Resources Association, Vol. 18(1), 95–99.

Artin, E. (1964). *'The Gamma Function'*, New York, Holt, Rinehart and Winston.

Bernard, M.M. (1935). 'Determination of flood flow by unit hydrograph method', *USGS Water Supply paper,* No. 771, 451–461.

Bhunya, P.K., Mishra, S.K. and Berndtsson, Ronny (2003). 'A simplified two-parameter gamma distribution for derivation of SUH', Journal of Hydrologic Engg., ASCE, 8(4), 226–230.

Bhunya, P.K., Mishra, S.K., Ojha, C.S.P. and Berndtsson, Ronny (2004). 'Parameter estimation of beta-distribution for unit hydrograph derivation', Journal of Hydrologic Engg., ASCE, 9(4), 325–332.

Chow, V.T. (1964). 'Handbook of applied hydrology', NY.

Clark, C.O. (1945). 'Storage and the unit hydrograph', Trans. of ASCE, 69:1919–1947.

Croley, T.E. (1980). 'Gamma synthetic hydrographs', *J. Hydrol.*, Amsterdam,47,41–52.

Dooge, J.C.I. (1959). 'A general theory of the unit hydrograph', J. of Geophysical Research, 64(2), 241–256.

CWC (1980). 'Flood estimation report for Lower Godavari sub-zone-3F', Design Office Rep. No. 3/1980, Directorate of hydrology (small catchments), CWC, New Delhi, India.

CWC (1982a). 'Flood estimation report for Lower Narmada and Tapi sub-zone-3B', Design Office Rep. No. M/5/1981, Directorate of hydrology (small catchments), CWC, New Delhi, India.

CWC (1982b). 'Flood estimation report for Mahanadi (sub-zone-3 D)', Design Office Rep. No. LNT/4/81, Directorate of hydrology (small catchments), CWC, New Delhi, India.

Edson, C.G. (1951). 'Parameters for relating unit hydrograph to watershed characteristics', Trans. Am. Geophys. Union, 32:591–596.

Espey, W.H Jr. and Winslow, D.E. (1974). 'Urban flood frequency characteristics' Proc. Of ASCE HY2, 179–293.

Flood Estimation Handbook, The Institute of Hydrology, Wallingford, Vol. 4, 1999.

Flood studies report, Vol. 1, National Environmental Research Council (NERC), UK, 1975.

Gray, D.M. (1961). 'Synthetic hydrographs for small drainage areas', Proc. of ASCE, 87, HY4, 33–54.

Gupta, V.K, Waymire, Ed and Wang, C.T. (1980). 'A representation of an instantaneous unit hydrograph from geomorphology', *Water Resour. Res.,* 16(5), 855–862.

Haan, C.T., Johnson, H.P. and Brakensiek, D.L. (Eds.) (1984). 'Hydrologic modeling of small watersheds', ASAE, Mihigan, USA.

Haktanir, T. and Sezen, Nurullah (1990). 'Suitability of two-parameter gamma distribution and three-parameter beta distribution as synthetic hydrographs in Anatolia', *Hydrol. Sc. J.*, 35(2), 167–184.

Hall, M.J., Zaki, A.F. and Shahin, M.M.A. (2001). 'Regional analysis using the geomorpho-climatic instantaneous unit hydrograph', Hydrology and Earth System Sciences, EGS, 5(1), 93–102.

HEC-1 (1990). 'Flood hydrograph package: *User's manual*', US Army Corps of Engineers, Washington, DC.

HEC-DSS (1994). 'User's guide to utility manuals', Computer program document No. 45, US Army Corps of Engineers, Washington, DC.

Johnson, N.L. and Kotz, S. (1973). 'Distributions in statistics: continuous univariate distribution –II', Wiley, NY.

Jueyi Sui (2005). 'Estimation of design flood hydrograph for an ungauged watershed', 19(6), 813–830, J of Water Res. Man., UK.

Kumar, R., Chatterjee, C., Lohani, A.K., Kumar, S. and Singh, R.D. (2005). 'Sensitivity Analysis of the GIUH based Clark Model for a Catchment', 16(4), 263–278, J of Water Res. Man., UK.

Kreyszig, Erwin (1993). 'Advanced engineering Mathematics', John Willey & Sons, Inc., Singapore.

Kull, D.W. and Feldman, A.D. (1998). 'Evolution of Clark's unit hydrograph method to spatially distributed runoff', Journal of Hydrologic Engg., ASCE, 3(1), 9–19.

Langhaar, H.L. (1951). 'Dimensional analysis and theory of models', Wiley, New York.

Linsley, R.K. Jr., Kohler, M.A. and Paulhus, J.L.H. (1975). 'Hydrology for Engineers', Second Ed., McGraw-Hill, New York.

Lohani, A.K. and Singh, R.D. (2001). 'Comparison of geomorphological based rainfall-runoff models', Technical Report, National Institute of Hydrology, Roorkee.

Maier, H.R. and Dandy, G.C. (2000). 'Neural networks for the prediction and forecasting of water resources variables: A review of modelling issues and applications'. Environmental Modelling & Software, Elsevier, 15, 101–124.

Marquardt, D.W. (1962). 'An algorithm for least-square estimation of nonlinear parameters', J. Soc. Indust. Appl. Math, Vol. 11, No. 2.

McCarthy, G.T. (1938). 'The unit hydrograph and flood routing', *Proc., Conf. of North Atlantic Div.,* Corps of Engrs.

Mendenhall, W. and Sincich, T. (1988). 'Statistics for the engineering and the computer sciences', Dellen Publ. Co., Scan Fransisco, Calif.

Nash, J.E. (1958). 'Determining runoff from rainfall", Proc. of the Institution of Civil Engg., 10:163–84, Dublin, Ireland.

Nash, J.E. (1959). 'Synthetic determination of unit hydrograph parameters,' J. of Geophysics Res., 64(1), 111–115.

Olivera, F. (2002). 'Extracting hydrologic information from spatial data for HMS modeling', J. Hydrologic Engg. ASCE, 6(6), 524–530.

Ponce, V.M. (1989). '*Engineering Hydrology*', Prentice Hall, NY.

Appendices

Appendix A

Assuming Eqn. 8 represents a UH of unit duration, the peak condition occurring at $t = t_p$ can be given by $dq_p/dt = 0$. Eqn. 8 can therefore be written as follows,

$$\frac{dq}{dt} = \frac{1}{B(r,p-r)}\left[(r-1)(t_p)^{r-2}\frac{(b-t_p)^{p-r-1}}{(b)^{p-1}} + \frac{(p-r-1)(b-t_p)^{p-r-2}(t_p)^{r-1}(-1)}{(b)^{p-1}}\right] = 0 \quad \text{... (a1)}$$

Thus,

$$(r-1)(t_p)^{r-2}\frac{(b-t_p)^{p-r-1}}{(b)^{p-1}} = \frac{(p-r-1)(b-t_p)^{p-r-2}(t_p)^{r-1}}{(b)^{p-1}}$$

Or $(r-1)(b-t_p) = (p-r-1)\, t_p$ or $t_p = \frac{b(1-r)}{(2-p)}$ ... (a2)

It follows from Eqns. (8) and (a2) that,

$$q_p = \frac{1}{B(r,p-r)}\left(b\frac{(1-r)}{(2-p)}\right)^{r-1}\left(1-\frac{b(1-r)}{2-p}\right)^{p-r-1} \quad \text{... (a3)}$$

At, $t = t_B$, $q = 0$. Therefore, from Eqn. 8,

$$t_B^{\ r-1} = 0$$

or $\frac{(b-t_B)^{p-r-1}}{(p-r)b^{p-1}} = 0$ or $b = t_B$ ... (a4)

Denoting $\alpha = t_B/t_p$ in Eqn. (a2),

$$r = 1 - \frac{(2-p)}{\alpha} \quad \text{... (a5)}$$

Appendix B

Conditions at peak of the hydrograph (refer Eqn. 8),

$$q_p = \frac{t_p^{\ r-1}(t_B-t_p)^{p-r-1}}{B(r,p-r)t_B^{\ p-1}} = \frac{t_p^{\ p-1}(t_p)^{-1}(\alpha-1)^{p-r-1}}{B(r,p-r)t_B^{\ p-1}} \quad \text{... (b1)}$$

$$\Rightarrow q_p t_p = \beta = \frac{(\alpha-1)^{p-r-1}}{\alpha^{p-1}B(r,p-r)} \quad \text{... (b2)}$$

Substituting the value of 'p' in terms of r (from Eqn. a5),

$$\beta = \frac{(\alpha-1)^{(1-r)(1-\alpha)}}{(\alpha^{1-\alpha(1-r)})B[r,(2-r)-(1-r)\alpha]} \quad \text{... (b3)}$$

Appendix C

Using Eqn. (9a),

$p = 2 - \alpha(1-r)$; $p-1 = 1 - \alpha(1-r)$ ... (c1)

Considering first two terms of Eqn. 11, the beta function can be approximated as,

$$B(r,p-r) = \frac{\Gamma p \Gamma r}{\Gamma(p+r)} \cong \frac{r^r(p-r)^{(p-r)}}{p^p}\frac{e^{-r}e^{-(p-r)}}{e^{-p}}\frac{(1+\frac{1}{12r})(1+\frac{1}{12(p-r)})}{(1+\frac{1}{12p})} \quad \text{... (c2)}$$

On simplifying and rearranging the terms Eqn. (c2) can be written as follows,

$$B(r,p-r) \cong 5/2\frac{\sqrt{r+1/6}\sqrt{13-6\alpha+6\alpha r-6r}}{\sqrt{13-6\alpha+6\alpha r}} r^{r-1}(2-\alpha+\alpha r-r)^{(r-1)(\alpha-1)}(2-\alpha+\alpha r)^{(\alpha-1-\alpha r)} \quad \text{... (c3)}$$