International Conference "Water, Environment, Energy and Society" (WEES–2009)
New Delhi, 12–16 January 2009

Statistics of Extremes in Hydrology

W.G. Strupczewski[1] and K. Kochanek[2]
Water Resources Department
Institute of Geophysics, Polish Academy of Sciences
Ksiecia Janusza 64, 01-452 Warsaw, POLAND
E-mail: [1]wgs@igf.edu.pl; [2]kochanek@igf.edu.pl

V.P. Singh
Department of Biological and Agricultural Engineering
Texas A&M University
Scoates Hall, 2117 TAMU, College Station, Texas 77843-2117, USA
E-mail: vsingh@tamu.edu

ABSTRACT: The main objections to the use of a pure statistical approach in the analysis of hydrological extremes are small sample size and unknown distribution function. The ML estimates of large quantiles are highly sensitive to the distributional choice, while the power of discrimination procedures is unacceptably low for hydrological sample sizes. The *L*-moments method seems to be the best for this purpose. Application of heavy-tailed distributions for extremes modelling is discussed. Moreover two-shape parameter distributions, while some of them are heavy-tailed, are proposed. Keeping in mind that the largest sample element is a low quality data, the effect of its omission on the *L*-moments accuracy of upper quantiles of two-parameter heavy-tailed distribution is examined. Recent developments in the statistics of extremes are primarily related to the maximum likelihood estimation in the presence of covariates. Its present and prospective hydrological applications are discussed with emphasis on non-stationary flood frequency analysis. As an alternative a two level estimation technique is proposed for estimation of non-stationary parameters of the distribution.

Keywords: Bias, Covariates, Discrimination Procedures, Heavy-tail, Hydrological Extremes, Non-stationary, Parameter Estimation, Probability Distribution, Quantiles.

INTRODUCTION

Statistics of extremes have played an important role in engineering practice for water resources design and management. The classical extreme value theory is built on the assumption that observations in the time series are Independent and Identically Distributed (IID). The cornerstone of this theory is the "three types" of distributions which can arise as limiting distributions of extremes in random samples, i.e., Gumbel, Fréchet and Weibull. They are combined into the Generalized Extreme Value (GEV) distribution, which has been widely used for modeling the distribution of flood peaks in at-site and regional settings. The main objection to this is that hydrological processes rarely produce observations that are IID. It opens the room for using alternative distribution families if they fit the data better.

In FFA, a Probability Density Function (PDF) is selected more or less subjectively from among positively skewed PDFs of continuous type. Some of these distributions were introduced because of their suitability to modeling different shapes of histogram or perhaps simply because they had not been used already (Cunnane, 1989). Few of them are supported on the basis of deductive reasoning about the genesis of floods. Since the theoretical arguments supplied for the purpose can be easily undermined, empirical suitability plays a much larger role in distribution choice than a priori reasoning. Obviously, the effect of the model selection is more pronounced and critical in the upper tail of a distribution.

Nowadays there is a growing consensus that hydrological extremes are heavy-tail distributed which is inherited from presumably heavy-tailed maximum precipitation. The present-day views on the causes of the appearance of heavy-tailed (inverse-power) distributions in nature are shortly presented in Section 2. Hydrological records are too short to provide sufficient

[1]Conference speaker

evidence for heavy-tailed property of extremes. The evidence got by "regional averaging" of distribution parameters of annual hydrological maxima (Hosking and Wallis, 1997) (i.e., a trade-off between space and time) can be undermined as an artifact of using the *L*-moments method (Klemeš, 2000). Hence, a variety of both thin and thick tailed distributions are equal as alternative models for a given sample. Our findings on the selection of the model that best fits a set of observations are shortly presented in Section 3. The possibility of correct identification of a PDF in the case of normal hydrological sizes of samples is small even in the ideal case when the set of alternative PDFs contains the true (*T*) distribution function. Therefore, in reality, one deals with the hypothetical PDF (*H*), called here the false distribution function (*F*), which differs more or less from the true one. This will result in a model error (bias) in any statistical characteristic of the distribution (Section 4). Its magnitude for a given characteristic depends not only on how closely is *F* to *T* but on the estimation method as well. In practice to assess the accuracy of estimation, the chosen model is considered as the true (*T*) one, i.e., the standard errors (*SE*) of estimated quantiles serve as the total error. However, an asymptotic bias caused by the false distributional assumption can be several times larger than *SE*. Section 5 discusses consequences on hydrological design of using heavy-tailed distributions. While scrutinizing annual flow maxima or the peaks over threshold, the largest element in a sample are often suspected to be low quality data, outliers or values corresponding to much longer return periods than the observation period. Section 6 deals with the effect of omission of the largest sample element on the accuracy of large quantile estimation obtained for two-parameter heavy-tailed distributions by the *L*-moments techniques. Introduction of the second shape parameter instead of the lower bound parameter is the subject of Section 7. Section 8 discusses the inclusion of covariates into probability distribution by making the distribution parameters functions of covariates. Special attention is paid to time-trend inclusion in flood frequency modelling (Section 9). Finally, the problem of multivariate hydrologic modelling is outlined in Section 10, and Section 11 concludes the paper.

CAUSES OF INVERSE-POWER DISTRIBUTION IN NATURE

Many geophysical phenomena, because of their complexity, manifest non-regular and chaotic behaviour. It appears, however, that statistically some observational distributions and patterns reveal that their nature is not purely random. The patterns have a fractal or multifractal structure and distributions resemble long-tail inverse power form. The observations include (after Czechowski, 2001): streamflows, topography, river networks, precipitations, rock fragments, mineral deposits, clouds, turbulent eddies, crack populations, fault distributions and magnitude frequency. The revelation of these facts is quite surprising—it testifies to some universality of behaviour of complex systems irrespectively of what processes (physical, biological or economical) they describe. Widespread appearance of inverse-power distributions in nature and human activities put self-evident questions about the reasons. The cause of inverse-power behaviour can be explained both by the non-linear approach (e.g., Czechowski, 2001) and the privilege approach (Czechowski, 2005). There is a relevance between these two approaches.

In the first approach, the structure of the medium, or the behaviour of intrinsic processes, is purely random on the lowest description levels, i.e., it may be characterized by purely random distributions such as: Poisson, exponential or Gaussian. Physical phenomena are modelled as a kind of black box g that transforms input variables x into an output variable $y = g(\mathrm{x})$ that is of interest in given phenomena. Unknown parameters or the intrinsic structure of the model is used as the input random variables. An amazingly wide class of non-linear models transforms random (exponential) distributions into a long tail distributions. The class includes increasing functions between power one ($g(x) = x^k$ for sufficiently large k) and those that increase very fast along a vertical asymptote. In the case of more non-linear functions $g(x)$ the inverse power forms of distribution functions appears for sufficiently large values of y. When the models are represented by differential equations, the degree of non-linearity of equations may be lower. Chaotic phenomena are caused by non-linearities of the model, and therefore non-linearities can lead to inverse-power distributions (see McCauley, 1995).

In the second approach, the inverse-power behaviour was derived using the privilege concept (where the privilege means the susceptibility of the state of the system onto a change). Long tails mean an excess of large events in comparison with purely random distributions as the Poisson, exponential or normal. This suggests that during the evolution of the system, large events are in some way privileged. In a physical planetesimal coagulation model, large bodies are privileged because they attract other stronger bodies by gravitational force. In the geometrical percolation model, larger clusters are privileged because they have longer perimeter and therefore grow faster (the geometrical privilege). In order to take into account the privilege, the model based on the master equation for the pure birth processes has been used with continuous time and discrete state space (N = 1, 2, ...). According to the form of the privilege function $B(N)$, one can obtain various forms of solution. For natural boundary conditions, inverse-power appears for $B(N) = N^{\alpha}$ when $\alpha > 1$. However, introducing the boundary conditions of the source type, the inverse-power solutions (steady state solutions) are obtained even for $\alpha > 0$ (i.e., even for a very weak privilege).

If the inverse power forms of distribution functions appear for large values of an event, a heavy tail is likely to hold for a distribution of annual extremes of this process. Morrison and Smith (2002) relaxed this condition showing that a mixture of Gumbel distributions (i.e., thin-tailed distributions) of the two independent random variables, as might arise when extremal distributions depend on the origin of flood, can resemble the GEV.

MODEL SELECTION

The selection of a correct or best-fitting distribution can have a significant effect on the reliability of flood-related structures. Even if the sample size is not sufficiently large for making a correct selection with high probability, a method of selection is still required and whatever information is available it needs to be utilized. In general, even though two models may exhibit similar fits to given data, it is, nonetheless, desirable for *FFA* to select the true (or more nearly correct) model, since inferences based on the model will involve tail probabilities where the effect of the model selection is more pronounced and critical.

In practice, two statistical identification approaches are employed as a decision procedure of statistical model building or identification: (1) goodness-of-fit procedure or (2) discrimination procedure. The goodness-of-fit procedures test whether or not the assumed distribution (also called the model) does indeed fit the data to a specific degree of confidence. They are of little value for model selection for moderate or small size samples.

A discrimination procedure must define a test statistic as well as a decision rule indicating the action to be taken for each observed sample. Several test statistics employed for this purpose are modifications or extensions of standard goodness-of-fit tests. Having defined a discrimination procedure, one selects from the set of competing models the model that is, according to the decision rule, best fitted to the data. One can also prioritize all competing models according to the values of the selection criterion.

Five two-parameter probability distributions, Gamma (*Ga*), Lognormal (*LN*), Weibull (*We*), Convective Diffusion (*CD*) (Strupczewski *et al.*, 2001a), also called Inverse Gaussian distribution (e.g., Tweedie, 1957), and Gumbel (*Gu*), were considered as alternative models for the distribution of annual peak flow discharges of Polish rivers (Mitosek *et al.*, 2006); and three estimation methods, *MLM*, *MOM* and *LMM*, were used. Five test statistics were employed for this purpose. The studies, comprising 39 historical time series of 70 years length revealed: (a) small differences in the values of the criterion function of the distribution considered; (b) some impact of the estimation method on the model selection; and (c) quite a high influence of the observation period on the model selection. In particular the *CD* model was selected as the best model by the Likelihood Ratio (*LR*) criterion in 27 of 39 cases, i.e., with a selection rate of 0.69, while the *LN* model was selected by the *QK* statistics (Quesenberry & Kent, 1982) with a selection rate of 0.97.

To evaluate the performance of each procedure, simulation experiments were considered for the analyzed cases of choice among any two and four models, i.e., *Ga, LN, We* and *CD* (Mitosek *et al.*, 2006; Strupczewski *et al.*, 2005). The variation in the efficiency of the procedures was investigated for various pairs of the considered distributions for different values of the coefficient of variation and sample sizes. These studies showed that the use of a discrimination procedure without the knowledge of its performance for the considered distributions may lead to erroneous conclusions. Usually, one of the models is favoured by the discrimination procedures. This imbalance depends on the procedure, competing models, parameters and the sample size. An example is shown in Table 1, where the *LR* procedure gave the probability of correct selection (*PCS*) of the *CD* model twice as large as that of the *LN* model and the *PCS* rarely exceeded 50% when the *LN* sequences were generated. On the contrary, the *QK* procedure (Quesenberry and Kent, 1982) highly favoured *LN* over *CD* (not shown). This explains the reason for selecting *CD* by *LR* as the dominant model for Polish data, and *LN* if the *QK* procedure was applied. In fact, what is considered in hydrology as a large sample is indeed a small sample if the problem of distribution choice is of concern. The use of several model discrimination procedures, combined with the knowledge of their efficiency for a given case, seems to be a promising way to increase the efficiency of the model selection techniques in FFA. The studies on the efficiency of discrimination of the true model show that in the ideal case, i.e., if the competitive set of models contains the true PDF, the probabilities of correct selection are probably unacceptable for "hydrological" sample sizes, and samples of sizes $N \geq 100$ are usually required to give large levels for these probabilities of correct selection. The discrimination power decreases with a number of parameters to be estimated from a sample. Therefore, in reality, one deals with the hypothetical PDF (H), called here the false distribution function (F), which differs more or less from the true one. This produces a model error in any statistical characteristic of the distribution. Its magnitude for a given characteristic depends not only on how closely is F to T but on the estimation method as well.

Table 1: Probability of correct selection for the *LR* procedure got by sampling experiment. Convective diffusion (*CD*) vs. Lognormal (*LN*) models—parameters estimated by MLM. Legend: N – sample size, C_V – variation coefficient

N	$C_v = .10$		$C_v = .25$		$C_v = .50$		$C_v = .75$		$C_v = 1.00$		$C_v = 1.50$	
	CD	*LN*	*CD*	*LN*	*CD*	*LN*	*CD*	*LN*	*CD*	*LN*	*CD*	*LN*
10	.59	.19	.62	.20	.69	.22	.73	.25	.76	.28	.80	.35
30	.63	.31	.65	.32	.69	.35	.74	.39	.77	.45	.80	.54
50	.66	.34	.67	.36	.70	.41	.75	.46	.79	.52	.83	.63
100	.70	.38	.71	.41	.71	.46	.76	.55	.80	.63	.88	.76
150	.81	.40	.81	.43	.81	.51	.80	.61	.82	.70	.92	.84

MODEL ERROR

In practice one deals with the "false" model and is not able to assess the magnitude of model error of any estimated statistics. To that end, the "true" PDF together with its parameter values should be known. The model bias of large quantile estimates depend on both the estimation method and sample size. The interest is to find a most robust estimation method for large quantiles to the false distributional assumption.

Strupczewski *et al.* (2002 a, b) analytically evaluated an asymptotic bias of four estimation methods caused by the assumption of a false probability distribution. The estimation methods were used by approximating the "true" model by the "false" one. Several pairs of two-parameter distributions bounded at zero showed that the asymptotic bias of large quantiles is an increasing function of the true value of the coefficient of variation (C_V), being smallest for *MOM* and largest for *MLM*. The bias of *LMM* occupied an intermediate position. Figure 1 shows the asymptotic relative bias of quantiles if the log-normal distribution asymptotic sample was falsely recognized as the log-Gumbel distributed. This finding essentially diminishes the practical usefulness of *MLM* in hydrological extremes analysis, because its efficiency may not compensate for the (frequently) huge bias produced by the assumption of a false *PDF* in the region of small exceedance probability quantiles the user is often interested in. It marks a departure of the hydrological extremes analysis from classic statistical theory of extremes of which the core is maximum likelihood method.

As shown in Figure 1, the upper quantile estimates got by the L-moment matching are much less sensitive to false distributional choice than ML estimates. Computational simplicity, small biases of sample estimates of L-moments and applicability for heavy-tailed distributions (namely, the L-moments exist whenever the mean of the distribution exists) all these give preference to the L-moments method in FFA.

Sampling bias can be smaller or greater than the model bias. It depends on the signs of its two components, i.e., estimation bias and model bias. In view of that one can doubt whether the work on the removal of the bias from estimation method for any particular distribution made under (H=T) assumption is important and relevant for FFM.

HEAVY-TAILED DISTRIBUTIONS IN HYDROLOGY

Wallis *et al.* (1974) assessed the dependence of the bias of sample Standard Deviation (SD) and skewness (C_S) on the distribution function, population skewness and sample size. Bias of both the SD and the C_S is negative and its absolute value grows with increasing C_S and sharply tends to zero with sample size. Extending their assessment for heavy-tailed distributions, we have found by simulation experiments that for two-parameter bounded at zero distributions, if the skewness coefficient is undefined, the sampling coefficient of variation is heavily underestimated and moreover the negative bias remains considerably high even for statistically large samples (Figure 2). It cannot be solely explained by the algebraic bound of C_V (Katsnelson and Kotz, 1957). Note that the algebraic bound depends on the sample size but not on the distribution and its population value of C_V. For a set of N non-negative values x_i, not all equal, the coefficient of variation C_V cannot exceed $(N-1)^{1/2}$, attaining this value if and only if all but one of x_i's are zero. Hence, for N = 10,000 one gets the upper bound $\hat{C}_V \leq 100$, while the largest population C_V considered equals two.

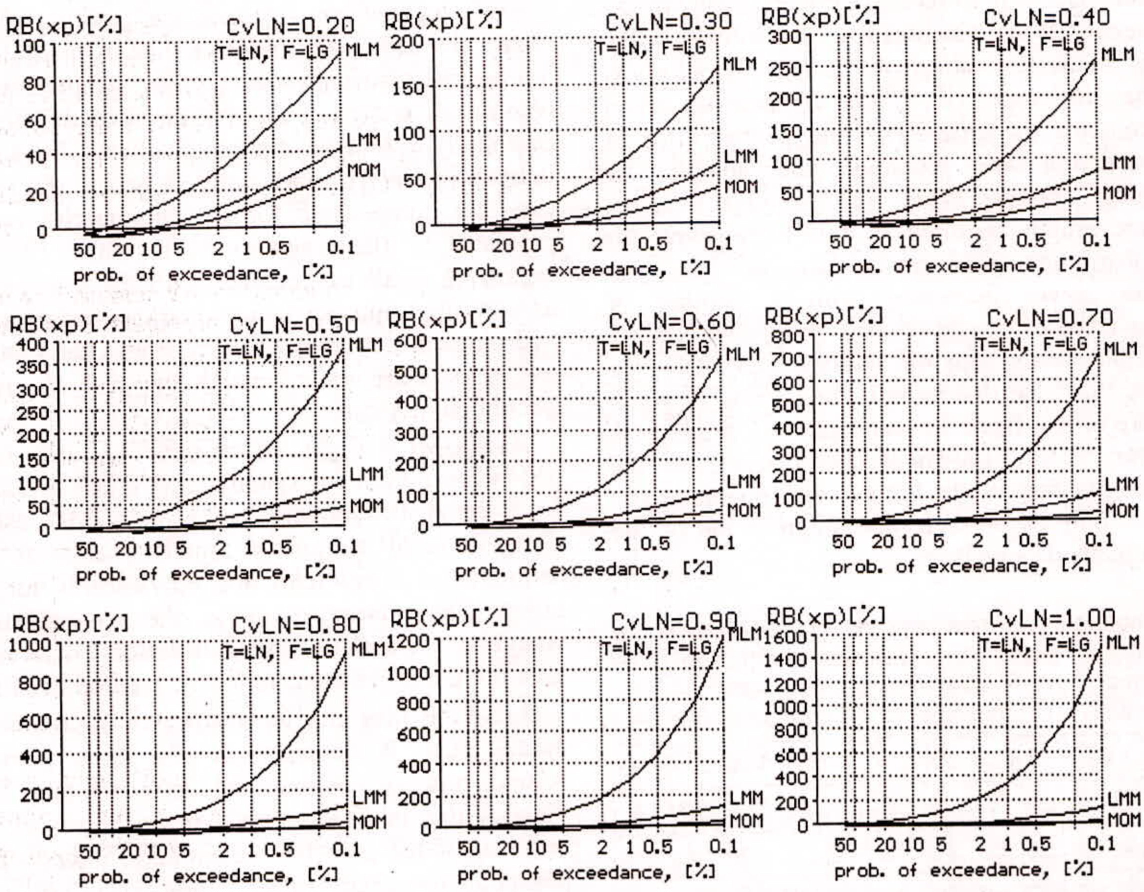

Fig. 1: Asymptotic bias of LG quantiles vs. probability of exceedance for some selected values of the true coefficient of variation C_VLN

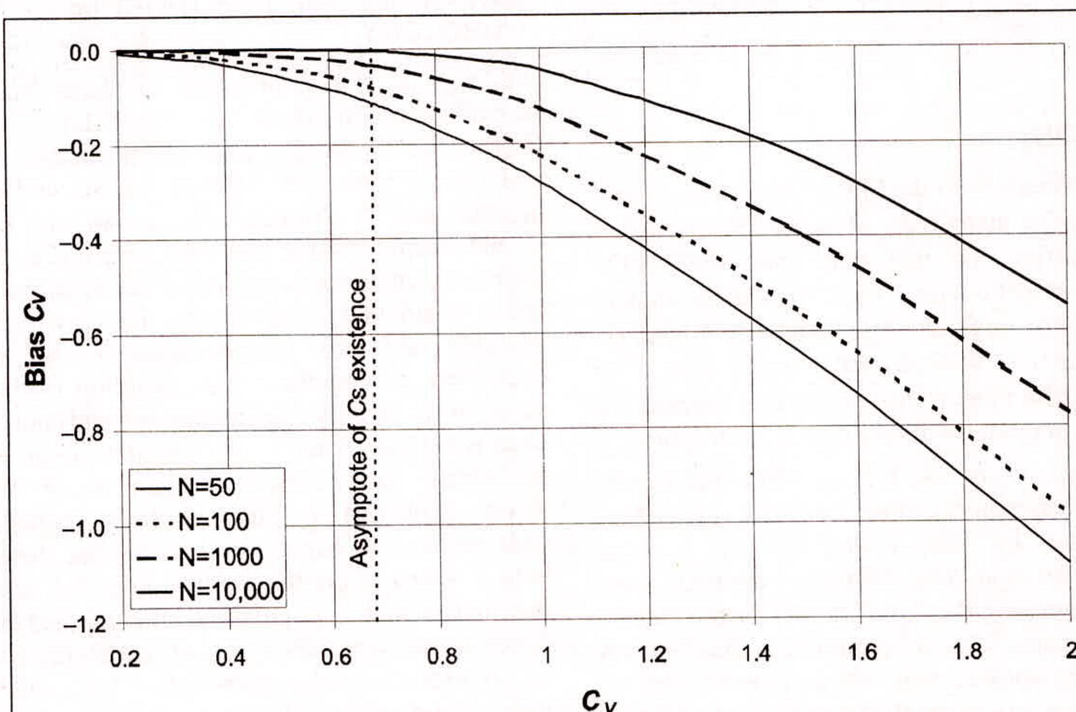

Fig. 2: Bias of MOM estimate of the variation coefficient for LG samples of various size (*N*)

Note that sampling moments are always finite, i.e., even if the respective population moments do not exist. Therefore, to be strict, allowance for heavy-tailed distributed samples ceases to apply MOM, unless one takes for granted that the population values of the matched moments are finite. In fact it is not easy to accept by hydrologists with engineering background that the inertia center or inertia moment of a figure may be undefined while quantiles are still finite values.

L-moment ratio diagrams for Polish rivers stations (Figures 3 and 4) do not point to heavy-tailed distributions as the best fitting models. The stations cluster around the log-normal and Gamma distributions. In general, lack of convinced arguments for preference in FFA of heavy-tailed models over thin-tailed models or vice versa gives a certain freedom for distribution selection.

Leaving the evidence for heavy tail of hydrologic extremes aside, one should evaluate practical importance of the problem, i.e., what consequences in terms of hydraulic design values arise from replacement of thin-tailed distributions by thick-tailed distributions. One can expect that whatever estimation method is used, upper quantiles of a heavy-tailed distribution will exceed the magnitude of the respective quantiles of any thin-tailed distribution matched to a sample. Values of two upper quantiles of seven two-parameter distributions, i.e., Log-Gumbel, Log-logistic, Log-normal, Gamma, Gumbel, Weibull and normal, and then five three-parameter distributions, i.e., Generalized Extreme Value, Generalized Log-Logistic (GLL), Log-normal, Pearson III type and Weibull, got by the *L*-moments and conventional moments methods, have been used for comparison. The results for the two-parameter distributions with mean equal to 100 are displayed for the *L*-moments method in Table 3 (the first *L*-moment: $\lambda_1 = 100$) and for *MOM* in Table 4 (the mean: $\alpha_1 = 100$). For the *L*-moments, the quantiles of LG and LL are greater than those of other distributions for any value of the coefficient of *L*-variation (τ). The differences grow with the τ value and the influence of thick tail is much more pronounced for greater cumulative probability (F) values. The relation of τ and CV is displayed in Figure 5.

The MOM estimates of quantiles (Table 4) are more robust to distributional choice than those of *L*-moments, which is also visible in Figure 1. The heavy-tail impact on upper quantile estimates is less noticeable here than using *L*-moments. Although for $C_V = 0.3$ and 0.6, the values of LG and LL are still the largest of all, for $C_V = 1.0$ and 1.5 both *LG* and *LL* produce for $F = 0.99$ lower values than all other distributions but normal. For $F = 0.999$, which corresponds to a major structure design value, and $C_V \leq 1$ the values of *LG* and *LL* are greatest of all.

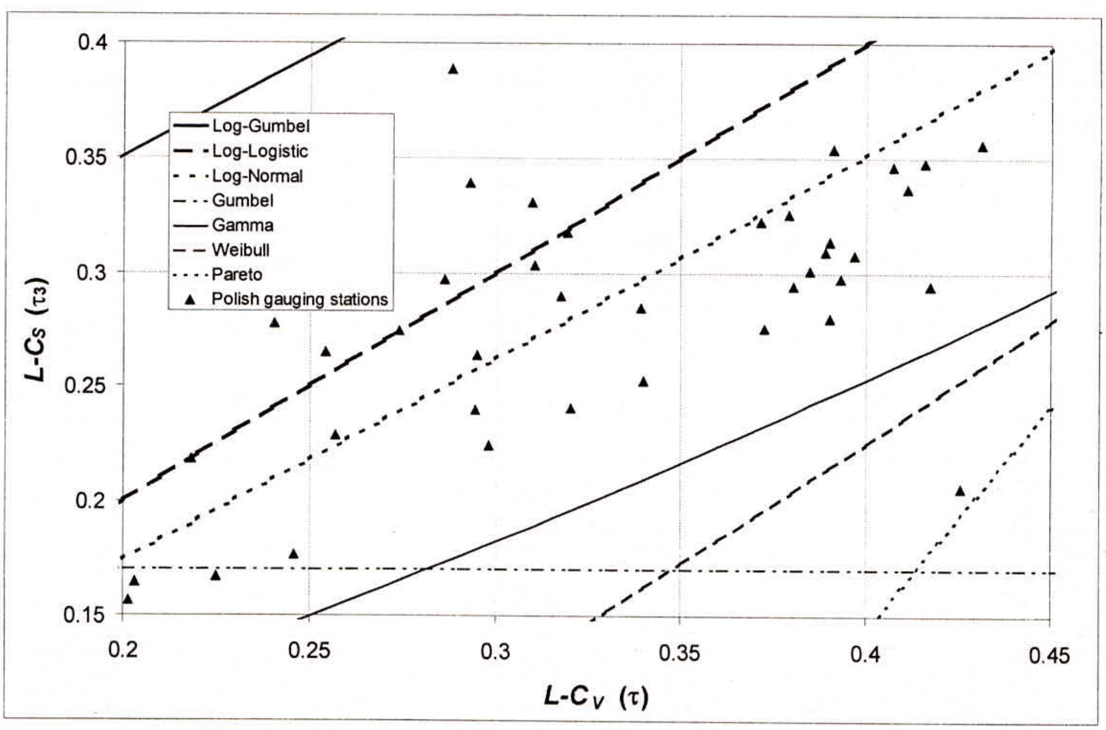

Fig. 3: *L*-moments ratios diagrams for 39 Polish rivers stations (two-parameter distributions)

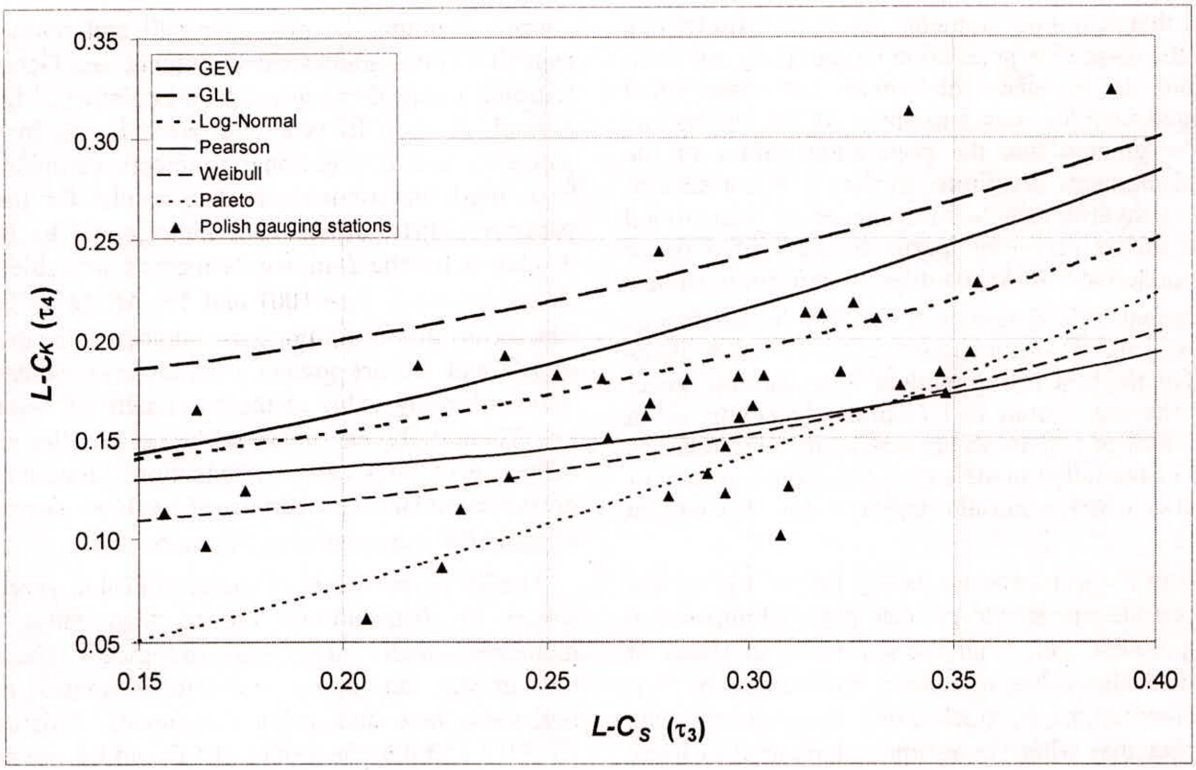

Fig. 4: *L*-moments ratios diagrams for 39 Polish rivers stations (three-parameter distributions)

Table 2: Upper Quantiles *vs.* the Coefficient of *L*-variation for Various Two-parameter Distributions

	$x_F = 0.99$				$x_F = 0.999$			
L-C$_V$ (τ) *PDF*	*0.1*	*0.2*	*0.3*	*0.35*	*0.1*	*0.2*	*0.3*	*0.35*
Log-Gumbel	160.9	270.7	395.0	467.1	206.4	497.8	944.8	1269.2
Log-Logistic	157.5	236.3	341.2	405.8	199.6	376.9	683.3	913.1
Log-Normal	149.1	215.8	307.1	360.5	170.9	283.7	466.3	587.2
Gumbel	159.6	216.1	275.7	303.9	193.8	282.6	376.4	420.8
Gamma	146.6	202.1	269.8	308.7	165.0	248.7	357.0	422.1
Weibull	135.8	182.1	248.9	289.8	144.5	207.3	307.1	373.3
Normal	141.9	183.8	223.3	244.2	155.6	211.3	263.8	291.6

Table 3: Upper Quantiles vs. the Coefficient of Variation for Various Two-parameter Distributions

	$x_F = 0.99$				$x_F = 0.999$			
C$_V$ *PDF*	*0.3*	*0.6*	*1*	*1.5*	*0.3*	*0.6*	*1*	*1.5*
Log-Gumbel	210.0	318.3	416.7	480.5	327.7	653.2	1037.3	1335.0
Log-Logistic	197.6	312.9	433.7	518.7	284.1	592.6	1022.5	1396.7
Log-Normal	189.6	311.5	490.5	693.3	237.3	475.8	926.5	1588.9
Gumbel	194.1	288.2	413.7	570.5	248.1	396.1	593.6	840.3
Gamma	182.7	289.0	460.5	707.7	218.7	388.9	690.8	1172.9
Weibull	167.1	272.9	460.5	719.2	186.4	345.6	690.8	1300.3
Normal	169.8	239.6	332.6	449.0	192.7	285.4	409.0	563.5

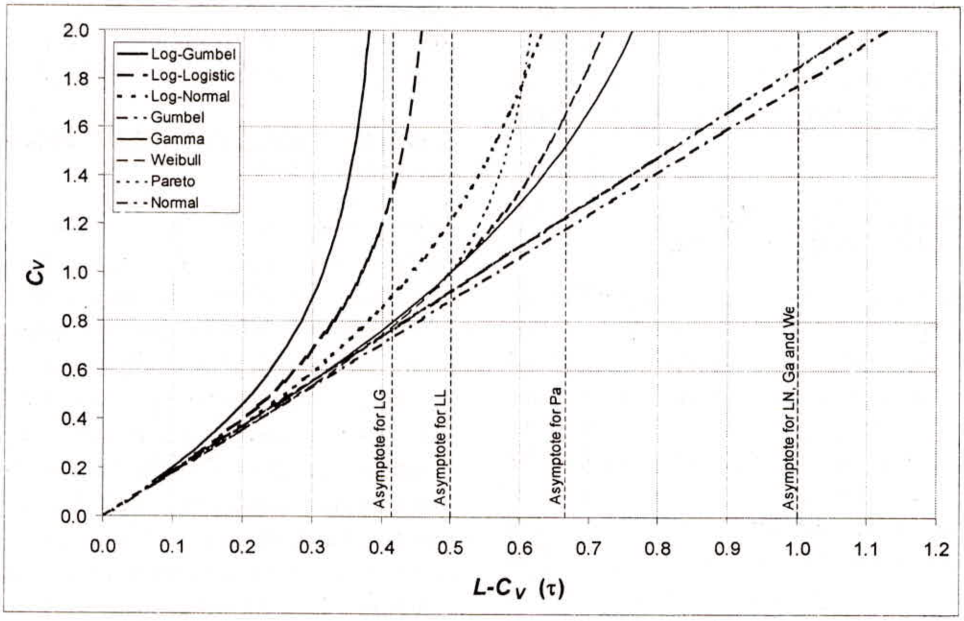

Fig. 5: The $CV-\tau$ relation for some two-parameter distributions

Table 4: Upper Quantiles vs. the Coefficient of *L*-skewness for Various Three-parameter Distributions

	$x_F = 0.99$					$x_F = 0.999$				
L-Cs (τ_3) *PDF*	*0.17*	*0.2*	*0.25*	*0.3*	*0.33*	*0.17*	*0.2*	*0.25*	*0.3*	*0.33*
GEV	5.80	6.21	6.93	7.70	8.17	9.13	10.37	12.86	15.96	18.14
GLL	6.36	6.73	7.36	8.02	8.43	12.25	13.62	16.25	19.39	21.55
Log-Normal	5.75	6.09	6.72	7.40	7.83	9.08	9.97	11.68	13.74	15.16
Pearson	5.58	5.85	6.34	6.85	7.17	8.38	8.96	9.96	11.04	11.74
Weibull	5.35	5.65	6.18	6.78	7.17	7.69	8.29	9.43	10.78	11.71

Table 5: Upper Quantiles vs. the Coefficient of Variation for Various Three-parameter Distributions

	$x_F = 0.99$					$x_F = 0.999$				
Cs *PDF*	*1.14*	*1.5*	*2*	*3*	*5*	*1.14*	*1.5*	*2*	*3*	*5*
GEV	3.14	3.31	3.48	3.64	3.71	4.95	5.58	6.33	7.34	8.14
GLL	3.09	3.23	3.42	3.46	3.54	5.49	6.02	6.97	7.23	7.96
Log-Normal	3.12	3.31	3.52	3.78	3.96	4.93	5.51	6.24	7.42	8.88
Pearson	3.11	3.33	3.61	4.05	4.57	4.73	5.23	5.91	7.15	9.22
Weibull	3.09	3.33	N.A.	N.A.	N.A.	4.55	5.13	N.A.	N.A.	N.A.

The results for the three-parameter distributions vs. the skewness are displayed for the *L*-moments method in Table 4 ($\lambda_1 = 0$; $\lambda_2 = 1$) and for *MOM* in Table 5 ($\alpha_1 = 0$; $SD = 1$). For *L*-moments, the GLL and GEV quantiles exceed the respective quantiles of thin-tailed distributions for any *L*-skewness coefficient value. The difference between quantile values amounts a few and tens percentage for $F = 0.99$ and $F = 0.999$, respectively.

Different ranking is noticed for MOM quantiles (Table 5), where for $F = 0.99$ and any value of the skewness coefficient, and for $F = 0.999$ but small skewness the values of GEV and GLL quantiles are less than those of thin-tailed distributions. Moreover, quantiles for a given C_S differ less than those of *L*-moments pointing to a greater robustness of MOM large quantiles to a distribution function. Note that GEV quantiles are greater than those of GLL, while quantiles from *L*-moments show an opposite order. The relation of τ_3 and C_S is displayed in Figure 5.

Concluding, replacement of thin-tailed models by thick-tailed models while using the *L*-moments estimation technique raise hydrologic design values, which may be particularly significant for major structure designs.

LARGEST SAMPLE ELEMENT OF HEAVY-TAILED DISTRIBUTED DATA

While scrutinizing annual flow maxima or peaks over a threshold, the largest elements in a sample are often suspected to be low quality data, outliers or values corresponding to much longer return periods than the observation period. Since, in the case of floods, the interest is focused mainly on the estimation of the right-hand tail of a distribution function, sensitivity of large quantiles to extreme elements of a series arises to the problem of special concern. It was investigated by simulation experiments using the *L*-moments method for both censored and complete samples generated from two-parameter heavy-tailed distributions, namely, from Log-Gumbel, Log-logistic, and Pareto. The results of simulation experiments show that omission of the largest sample element need not result in a decrease in the accuracy of large quantile estimates measured by RMSE (Kochanek *et al.*, 2007). Unfortunately, one has to bear in mind that this improvement occurs mostly at the expense of negative bias. Figure 7 shows for the LG quantile $x_{F=.99}$ got by the *L*-moments the ratio (*E*) of RMSE from censored sample applying the probability threshold value minimizing it to RMSE of complete sample, and Figure 8 the associated bias related to the population value of the quantile, i.e., $RB(\hat{x}_{.99}) = B(\hat{x}_{.99})/x_{.99}$.

DISTRIBUTIONS WITH TWO-SHAPE PARAMETERS

Taking into account the interest of flood frequency analysis in the right tail estimation and the doctrine of parameter parsimony, a replacement of the lower bound parameter by the second shape parameter seems to be advisable. The background and arguments for adding the second shape parameter as a replacement of the lower bound parameter are discussed by Strupczewski *et al.* (2007). In principle there are three ways of introducing the second shape parameter, i.e., power transformation of the variable (Tx), of its density function (Tf) and/or its cumulative distribution function (TF). Feasibility of such transformations in respect to commonly used two-parameter distributions is summarized in Table 6. Note that each of the three ways of introducing the second shape parameter is not feasible for every distribution. In some cases the transformation does not give the second shape parameter, i.e., after a conversion of transformed PDF one gets the initial distribution function (marked by "–"), in other cases the transformation is cumbersome (marked by "+ –"). The gamma Tx distribution is light-tailed for positive values of shape parameters and heavy-tailed for negative ones.

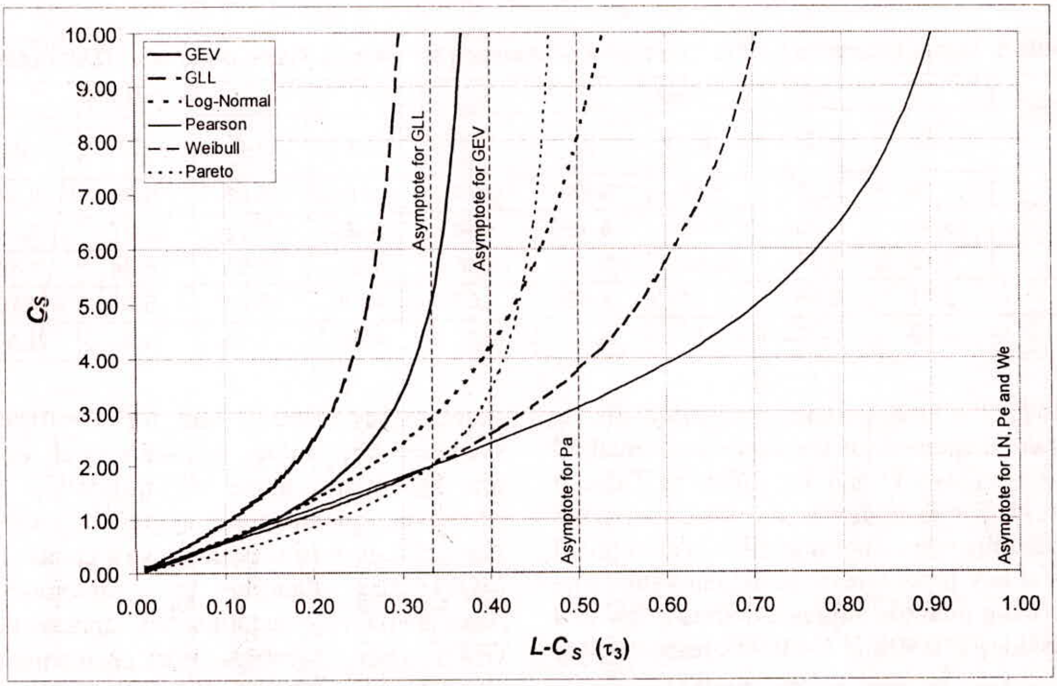

Fig. 6: The $CS–\tau_3$ relation for some three-parameter distributions

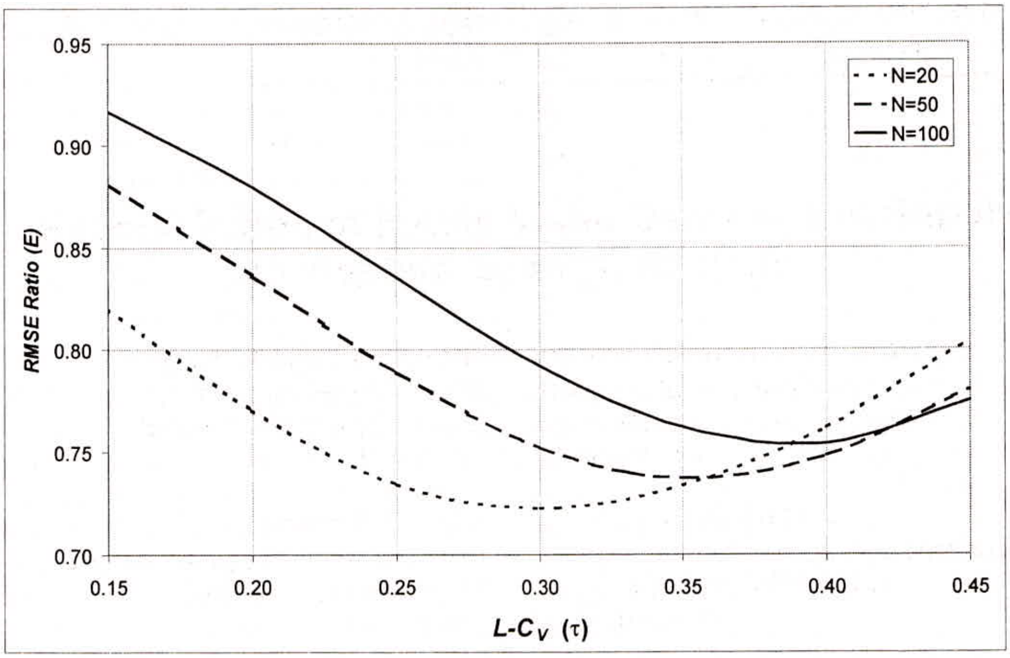

Fig. 7: RMSE Ratio vs. the coefficient of *L*-variation (τ) for $x_F = 0.99$

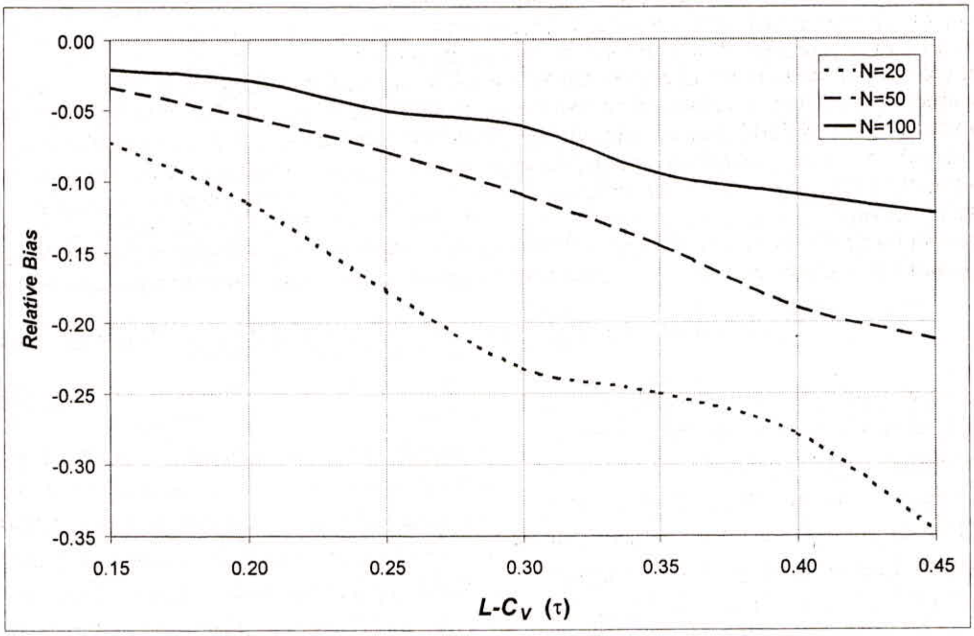

Fig. 8: Relative Bias of $x_F = 0.99$ from censored sample vs. coefficient of *L*-variation (τ)

Table 6: Feasibility of Second Shape Parameter Addition

Distribution Function	*Tx*	*Tf*	*TF*
Gamma	+	–	+ –
Weibull	–	+	–
Inverse Gaussian	+	+	+ –
Log-Normal	–	+	+ –
Log-Logistic	–	+	+
Log-Gumbel	–	+	–
Pareto	+	–	+

PROBABILITY DISTRIBUTION WITH COVARIATES

Recent developments as to the statistics of extremes are primarily related to the maximum likelihood estimation in the presence of covariates. Covariates can be either deterministic or random variables. They could incorporate trends, cycles, or actual physical variables (e.g., GCM indices) (Katz *et al.,* 2002). By fitting the extremal distribution conditional on the values of a covariate, the problem reduces to the ML estimation of covariate-dependent parameters. It has found application in non-stationary flood frequency analysis (Strupczewski *et al.*, 2001b, Katz *et al.*, 2002) and statistical downscaling of hydrologic extremes (Katz *et al.*, 2002) and may be useful to develop more rigorous statistical methodologies for regional analysis of extremes as well as in Bayesian methods (Katz *et al.*, 2002). In most cases, the fitting of covariate relationships has been based in least squares regression. Replacing conventional regression analysis techniques with the statistics of extremes can improve the rigor of hydrologic applications, such as trends and downscaling and make such analyses more physically meaningful and appealing.

The Points Over Threshold (POT) approach developed for independent events occurring several times a year is considered as a compromise between the annual maxima FFA and the classical time series analysis which is focused on modeling the auto-correlation structure of the time-series. Smith (2001) developed a statistical theory to apply the POT process approach to the statistics of extremes making allowance for covariates. The Poisson-Generalized Pareto model (i.e., the occurrence of exceedances and the excess over the threshold) is considered as a two-dimensional non-homogenous Poisson process. In this way, the GEV distribution can be indirectly fitted via the POT method, but still in terms of the GEV parameterization. The parameters of the GEV distribution could actually depend on time or other covariates. The basic idea of regional analysis is that the estimation of extreme quantiles at a given site can be improved by using extreme observations at the other side. There are some attempts (Buishland, 1991, Sveinsson *et al.*, 2001, Smith, 1989) to express regional analysis in terms of a formal statistical model including spatial dependence of extremes, i.e., parameters of the distribution dependent on location of a site within a given region.

To use the ML method in the presence of covariates, both probability distribution function and functional relations between its parameters and covariates should be defined. A larger number of parameters has to be estimated by the ML method if a covariate is included. More generally, the covariate could actually be a vector. However, one has to invoke the principle of parsimony in parameters. The number of parameters, which can be estimated reliably and efficiently from a hydrological sample which is of relatively small size, is very limited. Moreover, the ML is known to be extremely erratic for small samples (e.g., Kochanek *et al.*, 2005).

NON-STATIONARY FLOOD FREQUENCY ANALYSIS

The rejection of the stationary assumption with regard to the PDF parameters but not to its type reduces the problem to the stimation of time-dependent parameters ($\mathbf{h}_t$) of an assumed PDF: $f_t(x; \mathbf{h}_t)$. Three-parameter lower bounded distributions can serve as models. For the estimation of time-dependent parameters ($\mathbf{h}_t$) of a given PDF from observed time series ($x_1, x_2,\ldots, x_t,\ldots, x_T$), the assumption of continuous functional form of time dependence is needed, i.e., $\mathbf{h}_t = h(t, \theta)$ and $f(x; \theta, t)$. Following the doctrine of parameter parsimony, the smallest possible number of parameters θ is recommended. The linear function can be used to approximate trend in location and scale parameters, while the shape parameter can be considered constant. The probability distribution function with time-variant parameters is the preferable output of non-stationary FFA for further use in hydraulic design and water resources planning.

The model in NFFA means a type of probability distribution together with a class and form of time trend. In order to unify various PDFs in terms of parameters, the original set of parameters of each PDF would be replaced by the statistical moments using the relationships between moments and parameters available in the statistical literature: $[\mathbf{h}] \Rightarrow [m, \sigma, \gamma]$. The respective population moments are assumed to exist. Hence the reparameterized stationary PDF has statistical moments as parameters: $f(x; m, \sigma, \gamma)$. Consequently, the trend can be explicitly introduced in the mean and Standard Deviation, keeping the skewness constant (although non-stationarity in γ would be permissible as well), i.e., $f(x; m_t, \sigma_t, \gamma)$. Dealing with heavy-tailed distributions, one cannot be sure whether the population skewness exists. Therefore the original shape parameter may be left as it is. Introducing the trend function into PDF we get the vector of parameters $\theta = [\theta^{(m)}, \theta^{(\sigma)}, \gamma]$ of PDF $f(x; \theta, t)$, where γ stands for the skewness coefficient or for the original shape parameter.

Assuming a linear trend in the first two moments,

$$m_t = a^{(m)} + b^{(m)} \cdot t; \quad \sigma_t = a^{(\sigma)} + b^{(\sigma)} \cdot t \qquad \ldots (1a, b)$$

we get all together five parameters to be estimated from time series, i.e., $\theta^{(m)} = \left(a^{(m)}, b^{(m)}\right)$; $\theta^{(\sigma)} = \left(a^{(\sigma)}, b^{(\sigma)}\right)$ and γ. The number of parameters can be decreased by one if the trend is in the only one moment or the constant value of the coefficient of variation (C_V) is assumed. Other functional forms of trend in the moments or in original parameters could be considered. Therefore time-dependent quantiles are functions of cumulative probability F and time-dependent moments, i.e., $x(F, t) = x(F; m_t, \sigma_t, \gamma)$ or introducing the parameters of trend functions $x(F, t) = x(F, t; \theta^{(m)}, \theta^{(\sigma)}, \gamma)$. The $\hat{\theta}$ estimates are obtained from the condition,

$$\ln ML = \max_{\theta} \sum_{t=1}^{T} \ln f(x_t; t, \theta) \quad \text{... (2)}$$

Strupczewski *et al.* (2001) developed the identification of distribution and trend (IDT) package which serves to identify an optimum flood frequency model with time-dependent moments from a class of competing models. Originally it included six PDFs, namely, normal, two- and three-parameter log-normal, two- and three-parameter Pearson type III and Gumbel. Concerning the forms of trends, linear and parabolic options were included. Its extension for ten other two- and tree-parameter PDFs, namely, GEV, GLL, Inverse Gaussian, Inverse Gamma and Weibull, is in progress. For every model, from the time series the IDT software estimates model parameters by the ML method, derives the asymptotic covariance matrix of estimates and estimates for any given year or period the probability distribution of exceedances together with confidence intervals. The Akaike Information Criterion (*AIC*) is used to select the optimum model in a class of competing models $AIC = -2\ln ML + 2k$, where k is the number of independently adjusted parameters within the model. When several models for a given time series are available, the model that possesses the minimum value of *AIC* is considered as the most likely and it should be selected. For hydrological sizees of time series the *AIC* values of different models do not differ much and the number k highly influences the ranking of models.

Since the model error is not known, it is the standard error which is aliased with a measure of estimation accuracy. The approximate standard errors for the ML-estimated parameters and time dependent quantiles have been produced by the Fisher information matrix. It is quite difficult a task in case of complex multi-parameter distributions. Moreover as shown by simulation experiments, such standard errors can be quite unreliable for hydrological sample sizes.

"Resampling"—an alternative approach for determining standard errors is more handy than the information matrix approach and more universal as it is applicable for any estimation procedure. However it cannot be directly applied to nonstationary FFA as it requires IID data. Katz *et al.* (2002) overcame this difficulty by developing a multi-stage procedure. Making use of the fitted model the original time-series is converted to a sample consisting IID data. Using this sample, new samples are generated by bootstrap and then converted back by the inverse relation giving the bootstrap time-series. By refitting the model to a large number of bootstrap time-series and calculating the standard deviation, the standard error of desired statistics is obtained.

Replacement of original parameters by moments, aimed to bring closer FFA and trends in climatological time series investigation, discloses practical weaknesses of the ML method application to trend in the flood time series investigation. If the distribution is miss-specified, the ML-estimate of time-dependent moments will be biased and the estimate of trends as well its time. Therefore, different hypothetical distribution functions lead to trend estimates which may considerably differ, sometimes even in the sign of a trend. It is confusing as the clear-cut estimates are expected in climatological studies. It inclines towards an estimation of time-dependent moments by distribution-free techniques.

To deal with the trend in the two first moments, The Least Squares method has been generalized by Strupczewski and Kaczmarek (2001) to the situation where the assumption of constant variance does not hold and functional form of a trend in mean and standard deviation is given. Its generalization is the Weighted Least Squares (WLS) method where parameters of the trend in the mean and standard deviation are to be estimated simultaneously.

The WLS method does not require as rigorous a distribution assumption as does the ML method. The only assumptions are the existence of population moments upto the third order and the time invariable skewness. Note that the normal and Gumbel distributions have the constant skewness equal to zero and 1.14, respectively, while the skewness of three-parameter lower bounded distributions is defined by the shape parameter which can be assumed independent of time. Since the 'true' distribution functions of hydrological variables are not known, the restrictions do not seem to have much influence on the limitations of the WLS estimation in hydrology. The derived equations of WLS method are identical with those obtained for normal distribution by the ML method if trend in the both parameters are assumed. The AIC based on the normal distribution is employed to select the best fitting trend model. The four classes of time trends are analyzed: A. in the mean value; B. in the standard deviation; C. both in the mean and the standard deviation related by a constant value of the variation coefficient (C_V); D. unrelated trend in the mean value and the standard deviation. The basic option is the time-invariable parameters, called the stationary option (S).

The simulation experiments were applied to assess the performance of WLS estimates for non-normal distributions. The three-parameter Lognormal distribution with time-dependent mean and standard deviation and the constant skewness up to $C_S = 5$ was employed to generate time-series of different lengths. The bias of both trend estimates was found satisfactorily small. PCS of the trend model among A, B or C alternatives is high even for the hydrological size of time-series. However a longer time-series is required to discriminate the D-trend among A, B, C and D competitive models.

Having derived the time-variant mean $(\hat{m}_t)$ and standard deviation $(\hat{\sigma}_t)$ one can stationarize the time series $X = (X_1, X_2, ..X_t, ..X_T)$ by removing the trends from data,

$$Y_t = \frac{X_t - \hat{m}_t}{\hat{\sigma}_t} \quad \dots (3)$$

getting the series $Y = (Y_1, Y_2, .., Y_t, .., Y_T)$.

Subsequently stationary FFA can be performed on the Y data resulting in the quantile $\hat{Y}(F)$ estimates. To limit the influence of distributional choice on upper tail estimates, the L-moments method is advocated. Finally the time dependent quantile $\hat{X}(F;t)$ is obtained as,

$$\hat{X}(F;t) = \hat{Y}(F) \cdot \hat{\sigma}_t + \hat{m}_t \quad \dots (4)$$

Having estimated time-dependent parameters of the selected distribution $f(x;\hat{\theta},t)$, one can derive the cumulative probability for any single year. In hydrological design under non-stationary conditions the exceedance probability should refer to the whole period of life of a hydraulic structure. Denoting the service life as T-years and the beginning of operation in t_1 year, the probability of exceedance of peak discharge x_d during this period is,

$$P_T(X > x_d) = 1 - \prod_{t=t_1}^{t_1+T-1} \int_{-\infty}^{x_d} f(x;\hat{\theta},t)\,dx \quad \dots (5)$$

For a given probability of exceedance one can find by an iterative technique the design flow discharge x_d. Having asymptotic covariance matrix of parameter estimates $\mathrm{cov}(\hat{\theta}_i, \hat{\theta}_j)$ one can derive the standard error of $\hat{x}_d$.

Hydrologic design under non-stationary conditions is a direct consequence of accepting the idea of environmental changes. It requires a two-dimensional extrapolation, namely, in probability and in time, to cover the design life of a flood control structure, which can be over 100 years in the case of a major structure. One can wonder whether the statistical prediction for such a long period is reliable, i.e., whether the trend detected during, e.g., 50 years will last for the next 100 years. A physical explanation of the observed trend can make the prediction more meaningful. It is possible if a trend is a consequence of gradual change in land cover.

A special attention is paid to heavy floods and their time variability. Presumably heavy floods are generated by a different mechanism than small and medium floods. However the data are too short to implement it in FFA. Hence, in FFA the same distribution is assumed for all annual peaks of flow discharge. For a separate statistical analysis of heavy floods extraordinary long systematic records are required including historical flood information. The POT approach using the Poisson-Exp model with a high threshold q_C can be employed for the purpose. The assumption of independence of the number of events and their magnitude allows to estimate the time-dependent Poisson parameter and the exponential distribution parameter separately. The ML method was used by Strupczewski *et al.* (2001b) for the purpose.

MULTIVARIATE EXTREMES

Application of the statistics of multivariate extremes in hydrology is rather limited and it is mainly confined to bivariate extremes. A flood hydrograph is a very difficult object for stochastic modelling. Subjectively determining flood event starting and ending dates, and multiplying peaks make such characteristics as flood volume, time to peak or recession rate difficult to assess. A flood event as a multivariate event can be characterized by several variables like its peak, volume, duration, and time to peak. The usual practice in FFA is to consider a flood event as a univariate event and analyze separately flood peak or volume as a function of frequency. This is partly because a practical methodology for multivariate FFA is lacking due to the difficulty of deriving multivariate frequency distributions using conventional techniques. Each multivariate model must have the same marginal distributions. In practice, the two hydrologic variables may not have the same distribution and then their transformation is necessary. Except for the bivariate normal distribution, other bivariate models can hardly be extended to more than two dimensions as their correlation structure among variables is not known. Recent work on multi-dimensional copulas reported in the statistical literature may shed new lights on the afore-mentioned limitations. The concept of copulas is the two-step approach consisting of estimating the dependence function and the marginals separately. It allows to combine different types of mariginal distributions. A copula is such a tool which can be employed to derive multivariate distributions without the drawbacks of current multivariate distributions techniques. In FFA there is interest in copulas that emphasize correlation among extreme flood characteristics, i.e., in the right tails of the distributions. Several copulas that have this characteristic are offered in statistical literature and differences in shape among copulas can be discerned using descriptive functions.

The methodology for application of multivariate modelling in water resources management is not straightforward. Much useful to apply are conditional probability density functions based on the normal distribution. Analogously to the intensity-duration-frequency (IdF) model commonly used for rainfall analysis, the flood-duration-frequency model has been developed (e.g., NERC, 1975). It circumvents classical multivariate modelling. The object is the estimation of probability distribution of annual maximum mean discharge for the period of various durations (*d*). For each duration, a frequency distribution of maximum discharges is analysed and used to produce a continuous formulation of quantiles as a function of both probability and duration (Javelle *et al.*, 1999, 2002). This technique is worthy of special attention as not basing on the ML estimation in the presence of covariate (*d*), produces duration dependent location and scale parameters by the cost of the only one parameter, which is estimated separately from parameters of the standardized GEV distribution.

CONCLUSIONS AND RECOMMENDATIONS

The main objections to the use of a pure statistical approach in hydrological extremes analysis are small sample size and unknown distribution function. The hydrologist's interest is in the upper tail of the flood distribution, where various functions that fit the observed data satisfactorily may differ considerably and where estimates of extreme floods are unstable. Even if the true Probability Density Function (PDF) were known, it might, in all probability, contain too many parameters. These parameters cannot possibly be estimated reliably and efficiently from a sample of 'normal' hydrological size. Since no simple model can reproduce the dataset in its entire range of variability and the interest in FFA is in the estimation of upper quantiles, a statistical approach, based on the assumption of the known true frequency distribution function, falls short of accurately representing extreme hydrological events. The increasing mathematization of FFA has not increased the validity or accuracy of the estimation of high floods. The real potential for improving on standard extreme value techniques in hydrology comes not in finding estimates which improve slightly on existing ones, but in generalizing and adjusting the methods to handle richer sources of data (e.g., Katz *et al.*, 2002), i.e., the time series of daily flows, regional data and global atmospheric circulation. Such additional information is included into modeling by the cost of the assumptions concerning the model structure or relations between distribution parameters and covariates, and the number of parameters to be estimated. All these assumptions as well the functional relationships between parameters of extremal distributions and covariates should arise from physics. In general, the impossibility of 'true' model identification even if it is of simple form, sample constraints in multi-parameter estimation, the assessed magnitude of the model error of upper quantile estimation, the non-stationarity of river flow process, and problems of ungauged catchments lead to the conclusion that we should go back and start to work on the physics of extremes in hydrology. The hope is credited to the development and application the non-linear geophysical theory of floods in river networks (e.g., Gupta, 2004). This theory, called the scaling theory, has the explicit goal to link the physics of run-off generating processes with spatial power-law statistical relations between floods and drainage areas across multiple scales and time. Poveda *et al.* (2007) showed that power laws describe the relationship between annual flood quantiles and drainage areas and the flood scaling parameters can be expressed as functions of run-off obtained from water balance. Observed power laws in floods for individual rainfall-run-off events may be related to annual flood frequencies. The scaling theory of floods provides the scientific foundations for making flood predictions and FFA under a changing hydro-climate.

ACKNOWLEDGEMENTS

This work was partly supported by the Polish Ministry of Science and Higher Education under the Grant 2 P04D 057 29 entitled "Enhancement of statistical methods and techniques of flood events modelling".

REFERENCES

Buishland, T.A. (1991). "Extreme rainfall estimates by combining data from several sites". *Hydrol. Sc. J.*; 36, 345–65.

Cunnane, C. (1989). "*Statistical distributions for flood frequency analysis*". WMO-No. 718, Operational Hydrology Report No. 33, Geneva, Switzerland, pp. 73+6 Appendices.

Czechowski, Z. (2001). "Transformation of random distributions into power-like distributions due to nonlinearities: application to geophysical phenomena". *Geophys. J. Int.* 144, 197–205.

Czechowski, Z. (2005). "The importance of the privilege in resource redistribution models for appearance of inverse-power solutions". *Physica* A 345, 92–106.

Gupta, V.K. (2004). "Emergence of statistical scaling in floods on channel networks from complex run-off dynamic". *Chaos, Solitions and Fractals*, 19, 357–365.

Hosking, J.R.M. and Wallis, J.R. (1997). *Regional Frequency Analysis*, Cambridge Univ. Press, pp. 224.

Javelle, P., Grésilon, J.M. and Galéa, G. (1999). "Discharge-duration-frequency curves modeling for floods and scale invariance". *Comptes Rendus de l'Académie des Sciences. Sciences de la terre et des planets*, 329, 39–44.

Javelle, P., Ouarda, T., Lang, M., Bobée, B., Galéa, G. and Grésilon, J.M. (2002). "Development of regional flood-duration-frequency curves based on the index-flood method". *Journal of Hydrology*, 258, 249–259.

Katsnelson, J. and Kotz, S. (1957). "On the upper limit of some measures of variability". Archiv. Meteor, F., *Geophys. u. Bioklimat.*, (B), 8, 103.

Katz, R.W., Parlange, M.B. and Naveau, P. (2002) "Statistics of extremes in hydrology". *Adv. Water Resour.* 25, 1287–1304.

Klemeš, V. (2000), "Tall tales about tails of hydrological distributions". II. *J.of Hydrol.Eng.*, V.5. No.3. 232–239.

Kochanek, K., Strupczewski, W.G, Singh, V.P. and Weglarczyk, S. (2007) "Performance of the PWM large quantile estimates of LG distributed samples deprived of largest element". Submitted to *Hydrol. Sc. J.*

Kochanek, K., Strupczewski, W.G, Singh, V.P. and Weglarczyk, S. (2005). "Are parsimonious FF models more reliable than true ones? II. Comparative assessment of the performance of simple models versus the parent distributions". *Acta Geoph. Pol.*, 53(4), 437–457.

McCauley, J.L. (1995). "*Chaos, Dynamics and Fractals. An Algorithmic Approach to Deterministic Chaos*", Cambridge University Press, Cambridge.

Mitosek, H.T., Strupczewski, W.G. and Singh, V.P. (2006). "Three procedures for selection of annual flood peak distribution". *J. of Hydrol.*, 323, 1–4, 53–73.

Morrison, J.E. and Smith, J.A. (2002). "Stochastic modeling of flood peaks using the generalized extreme value distribution". *Water Resour. Res.*, 38, 12, 1305, 41 – 1–12, doi:1029/2001WR0-00502.

NERC (1975) *Flood studies report*. Nat. Env. Res. Council, London.

Poveda, G., Vélez, J.I., Mesa, O.J., Cuartas, A., Barco, J., Mantilla, R.I., Mejia, J.F., Hoyos, C.D., Ramirez, J.M., Ceballos, L.I., Zuluaga, M.D., Arias, P.A., Botero, B.A., Montoya, M.I., Giraldo, J.D. and Quevedo, D.I. (2007). "Linking long-term water balances and statistical scaling to estimate river flows along the drainage network of Colombia". *J. Hydrol. Eng.*12, 4–13.

Quesenberry, C.P. and Kent, J. (1982). "Selecting among probability distributions used in

Reliability". *Technometrics* 24(1), 59–65.

Smith, J.A. (1989). "Regional flood frequency analysis using extreme order statistics of the annual peak record". *Water Resour. Res.*, 25, 311–7.

Smith, J.A. (2001). "*Extreme value statistics in meteorology and the environment. Environmental statistics*". Chapter 8: 300–357 (NSF-BMS conference notes). Available at http://www.stat.unc.edu/postscript/rs/envstat/env.html.

Strupczewski, W.G. and Kaczmarek Z.,(2001). "Non-stationary approach to at-site flood-frequency modelling. Part II. Weighted least squares estimation". *J. Hydrol.*, 248, 143–151.

Strupczewski, W.G., Singh, V.P. and Weglarczyk, S., (2001a). "Impulse response of Linear Diffusion Analogy model as a flood frequency probability density function". *Hydrol. Sci. J.* 46(5), 761–780.

Strupczewski, W.G., Singh, V.P. and Feluch, W., (2001b). "Non-stationary approach to at-site flood-frequency modelling. Part I. Maximum likelihood estimation". J. Hydrol., 248, 123–142.

Strupczewski, W.G., Singh, V.P. and Weglarczyk, S. (2002a). "Asymptotic bias of estimation methods caused by the assumption of false probability distribution". *J. of Hydrol.* 258, 1–4, 122–148.

Strupczewski, W.G., Węglarczyk, S. and Singh, V.P. (2002b). "Model error in flood frequency estimation". *Acta Geophys. Pol.* 50, 2, 279–319.

Strupczewski, W.G., Mitosek, H.T., Kochanek, K., Singh, V.P. and Weglarczyk, S., (2005). "Probability of correct selection from Lognormal and Convective Diffusion models based on the likelihood ratio". *SERRA* 20: 252-163. DOI: 10.1007/s00477-005-0030-5.

Strupczewski, W.G., Markiewicz, I., Kochanek, K. and Singh, V.P. (2007). "Short Walk into Two-Shape Parameter Flood Frequency Distributions", in: Advances in Hydrology and Hydraulics, edited by V.P. Singh, *Water Resources Publications, Highlands Ranch,* in press.

Sveinson, O.G.B., Boes, B.C. and Salas, J.D. (2001). "Population index flood method for regional frequency analysis". *Water Resour. Res.* 37, 2733–48.

Tweedie, M.C.K. (1957). "Statistical properties of the inverse Gaussian distributions", *I. Ann. Math. Stat.*, 28, 362–377.

Wallis, J.R., Matalas, N.C. and Slack, J.R. (1974). "Just a moment!" *Water Resour. Res.* 10, 2, 211–219.