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FLOW TOWARDS WELL WITH STORAGE IN LEAKY AQUIFERS

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ABSTRACT

The formulation of strategy for the optimal management of groundwater demands a knowledge of the relationship between the pumpages in the wells to the drawdowns in the aquifers both in spatial and temporal coordinates. Such relationships are available from several analytical solutions in well hydraulics, the notable being Theis and Hantush solutions. However, they have been derived under highly idealised conditions. Thus there exists a need to refine the analytical solutions by incorporating the actual field conditions to the extent possible. Most of the available solutions consider the well to be a line sink and do not consider the storage in the well while most of the wells in the country are dug wells of large diameter with huge amount of storage in them. The application of the existing methods of analysis for the assessment of aquifer parameters or for modeling the aquifer or for simulating the drawdown history from the wells of this type would be erroenous. Hence in the present investigation, the simulation for the unsteady flow towards finite diameter wells storage has been attempted. The analysis includes the problems of wells with storage in leaky aquifers both for fully and partially screened wells.

The existing analytical solutions reveal that it would be impossible to incorporate the boundary conditions into the analytical formulation and bring out a solution that is tractable. Hence finite element method based on the variational

formulation of the flow problem has been used in the analysis. Appropriate sizes for the elements are chosen while discretizing the flow domain. Considering the accuracy demands of the problem, the Crank Nicolson scheme has been adopted for time discretization. Type curves have been presented for the case of fully and partially screened wells with storage for leaky aquifers for various storage parameters. The critical study of these curves indicated that there are three zones in each of the type curve. Zone-1 of the curve is a straightline with a steep slope indicating that the contribution of the aquifer storage for the discharge is negligible. Zone-3 of the type curve matches with the type curve for wells'with no storage' signifying the effects of well storage have practically vanished. The Zone-2 has been considered a transition between Zone-1 and Zone-3 which varies with the parameters that are indentified. The time of pumping corresponding to the end of zone-1 is termed as deviation time and that of zone-2 is defined as merging time. The storage effects are found to decrease both with time and radial distance from the pumping well but to increase with the storage parameter. The limits of storage effect zone have been identified. Ignoring the effects of storage in this zone would lead to significant errors. The drawdowns in the case of partially screened wells increase with decrease in screening ratio. The steady state conditions will be reached much earlier as it has been found that the effects of leakage and the storage of the well are cumulative in nature. From the concept of time of deviation a general expression for assessing the transmissibility of the aquifers

is presented. It has been found that the expression for time of deviation remains the same for Darcy flow whatever be the variation of the storage parameter and radial distance from the pumping well, while the merging time is found to vary with the radial distance.

1.0 INTRODUCTION

1.1 General

During the last decade, it has been more fully realized that refined quantitative answeres are needed in the evaluation of available resources. Competition for the available resources has brought about an awareness and as such, one of the principal problems of the Government is the resource management. Before the ground water resources can be managed, they must be quantitatively appraised. Proper planning and management of this important resource require the testing of all possible schemes and appraising of the relative merits of various alternatives. The problems related to the ground water resource management basically are the determination of sustaining yields of wells and aquifers, the interference between wells and well fields , the interrelation between surface and sub-surface waters and the quality of water.

Questions pertaining to the use of ground water resources require the establishment of proper relationship between pumpages and water level changes both in time and space and for doing so, the hydrogeologic properties of aquifers and aquitards, their dimensions in space and the boundaries are of utmost importance. This relationship cannot be established until hydrogeological maps are available which encompass all nonhomogeneous and irregular hydrogeological conditions. Thus, there is a necessity to determine the hydrogeological parameters like transmissibility, leakage factor and storativity etc., with reasonable accuracy. For these purposes, pumping or recovery tests are performed on the wells penetrating into the water bearing strata in the subsurface.

A water bearing stratum, called an aquifer may be confined, semiconfined or unconfined. A confined aquifer is a permeable bed confined in between two impermeable layers. A semiconfined aquifer, also called a leaky aquifer is the one which is bounded above and/or below by a semipervious layer having low but still measurable permeability. This semipervious layer is generally known as aquitard.

The wells, generally, are screened through the entire portion of the aquifer thickness. Such wells are termed as fully screened wells. However, in the event when the aquifer is too thick, the wells can be partially screened even. Most of the times, these partially screened wells are economical, as the cost of the well may be less without significant reduction in discharge. Also, the wells may be either open dug wells or tube wells. Dug wells are usually, large diameter wells (varying from 2 to 10 meters) of small depth. In some parts of India and in Asia, most of the existing wells are of this type. Generally, they will be either penetrating a leaky aquifer or a water table aquifer and at times penetrating into confined aquifers with a reduced well radius. A typical dug well in a confined aquifer is shown in Figure 1.

The flow towards a well may be steady or transient. The flow is said to be steady if the discharge from the well and the recharge of the aquifer to the well is equal. In this

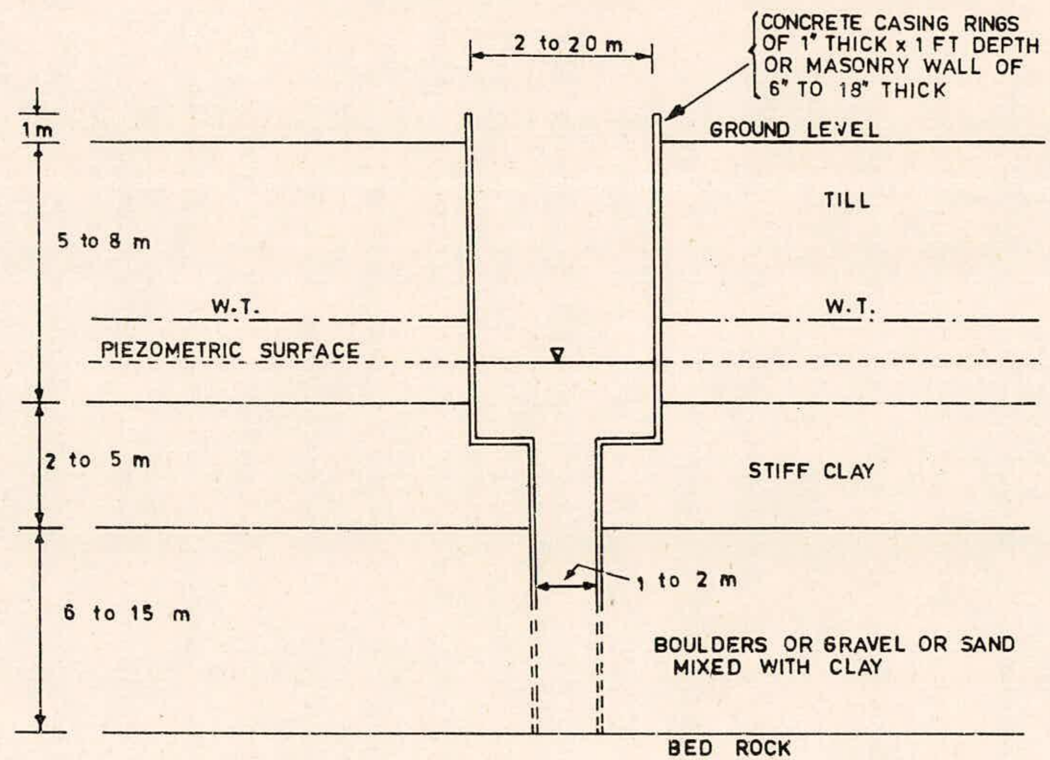

FIG. 1- A TYPICAL WELL WITH STORAGE

case, the discharge from the well and drawdown are invariable with respect of time. Transient or nonequilibrium flow occurs from the moment pumping starts till the steady state conditions is reached.

There have been a series of developments between 1856 and now in understanding the flow phenomena through porous media and the factors that govern this flow which has helped to establish the principles of ground water resources evaluation as a quantitative science. Darcy's law serves as basis for numerous quantitative methods in the field of ground water flow. The mathematical analysis for relating cause and effect available in literature, deals with the problems with simple boundary conditions. The complicated hydrogeological problems which generally occur in real life situations require certain simplifications for mathematical amenability. However, in most of the cases, such simplifications make the solutions unrealistic and may not yield a true relationship between cause and effect. In the recent past, due to the enormous demands on the ground water, a proper management policy over the pumping rates and other factors has become essential, for which accurate methods of determination of the aquifer properties are needed. This resulted in the necessity for the development of more accurate analysis for the complex ground water problems like finite diameter wells with storage, aquifers with anisotropy and/or inhomogeneity with respect to transmissivity and aquifer with irregular time variant boundaries etc.

In India the most of the existing wells are dug wells with large amounts of storage in them. The study conducted by Papadopulos and Cooper (1967) suggests that use of Theis curve/Hantush curves for the analysis of pumping test data obtained from the large diameter wells would yield erroneous aquifer parameters. Though, it is clear that the influence of well storage would effect the drawdown distribution in the aquifer, their study was limited to the well boundary only. In view of the proven necessity for a thorough study for assessing the proper drawdown/piezometric head distfibution within the aquifer when pumping tests are conducted in the wells with storage, a finite element model was developed by the author. This model was used to study the effect of well storage on the piezometric head distribution in the aquifer and for the establishment of type curves in leaky aquifer. Also, the zone of applicability of these type curves both in time and space was established. The present investigation deals with the study of wells with storage in semiconfined/ leaky aquifers and establishment of type curves which could be used for the proper assessment of hydrogeologic properties of aquifer-aquitard systems. Also an attempt is made to identify the region of the applicability of these type curves both in time and space.

1.2 Scope

It is thus noted that the studies made till now on the effects of well storage over the drawdowns around wells, partially and fully screened in leaky aquifers, have been very little. Accordingly, it is proposed in the present investigation, to make a comprehensive study of the effects of well storage on drawdown around wells mentioned above. For this study, the aquifer is taken to be homogeneous and isotropic and the flow is in transient state. It is proposed to employ the finite element techniques for the analysis of the problem in view of the several advantages offered by this method.

2.0 REVIEW

Exploration of underground natural resources is highly dependent on the development of well hydraulics. The usage and importance of groundwater is known from as early as 1000 B.C. (Keilhack,1912). However, it is only after the advent of Darcy's law in 1856, the rigorous and sound mathematical approach could be given to the subsurface flow. During the last two to three decades important developments have contributed much to such understanding by reassessing the basic physical principles which determine the behaviour of water flowing through porous media. Advances in mathematics and computer technology have facilitated the analysis of complex problems in the field of groundwater hydraulics.

2.1 General Field Equation for Transient Seepage Flow

Combining the Darcy's law and the law of conservation of mass, popularly known as continuity equation, the general field equation in the three dimensional cartesian coordinate system for the case of transient flow through porous media of uniform thickness can be obtained as (Walton,1970).

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{S}{T} \cdot \frac{\partial s}{\partial t} \qquad \ldots.(1)$$

where, s = drawdown in the well,

x,y,z = coordinates axes,

S = storage coefficient

t = time of pumping from the start

T is defined as the transmissibility of the medium and is equal to the product of the thickness of the aquifer 'm' and the Darcy's permeability Coefficient 'K'

i.e. $T = K \times m$

The equation 1 can be rewritten in radial coordinates as

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial z^2} = \frac{S}{T} \cdot \frac{\partial s}{\partial t} \qquad \ldots(2)$$

Equation 1 is of the same form as the fundamental equation of applied physics known as diffusion equation. Under steady state conditions of flow, the right hand side of the equations 1 and 2 become zero, since the velocity and therefore the pressure distribution is invariant with respect to time.

2.2 Flow towards a Fully screened well without storage in a confined aquifer

Dupuit (1863) obtained an expression for the steady state flow to a fully penetrating well in confined aquifer. However, for the usage of Dupuit's equation, the knowledge of radius of influence is essential. Thiem (1870) reviewed Darcy's experiments and derived an equation similar to that of Dupuit's interrelating the piezometric surfaces at any two known radial distances with discharge from the well. Using this equation, the transmissibility of the aquifer can be obtained. But in a practical case, it is difficult to obtain a steady state condition, which makes the applicability of these equations very much limited. Attempts have been made subsequently, to use the transient history of the pumping of the well to determine the aquifer properties. Thus, Theis (1935) presented a solution to the diffusion equation(Eq.1) from the analogous solution of the heat conduction theory. In his analysis, the production well is replaced by a mathematical sink of constant strength and is assumed to have an infinitesimal diameter. Subsequently Hantush(1964) considered the production well as a finite diameter well instead of a line sink and obtained an exact solution for the drawdown equation.

2.3 Flow towards a Fully Screened well with storage in a confined aquifer

Since in majority of middle Eastern and Asian countries most of the wells are dug wells, the above mentioned tables or type curves given either by Hantush or Theis are not suitable as they do not consider the relatively large capacity of the pumping well itself. Papadopulos and Cooper Jr.(1967) attempted to give an exact solution for the drawdown in finite diameter wells with storage. They have suitably modified the boundary condition on the well face expressing the rate of discharge pumped from the well as equal to the sum of the rate of flow of water into the well from the aquifer and the rate of decrease in the volume of water within the well.

$$\text{i.e. } Q = Q_W + Q_A \qquad \ldots(3)$$

where,

Q = the water pumped from the large diameter well

Q_W = the water contributed by the well storage

Q_A = the water contributed by the aquifer

The general differential equation(equation 1) for unsteady radial flow with suitable initial and boundary conditions was solved by them using Laplace transform technique and an expression for the drawdown in the well was obtained.

Fenske (1977) also obtained an expression for the drawdown around large diameter well with storage. His analysis includes effects of storage of the pumping well as well as the observation well. He obtained the necessary equations by a simple modification of the Theis equation. His analysis assumes that the water stored in the observation well would recharge the aquifer instantaneously with a drop in the head in the adjoining aquifer. Since the rate of recharge from the observation well depends upon the aquifer parameters and differential head between the aquifer and the well, significant error may exist in the early time portion of the drawdown episode.

2.4 Flow towards a fully screened well without storage in Leaky Aquifers

The confining beds of an artesian aquifer are rarely completely impermeable. Frequently, the artesian sand is confined above and/or below by semi pervious elastic clay or silt that yield significant amounts of water from storage. Hantush (1956) gave equations for the drawdown in the wells in leaky artesian aquifers. The governing differential equation for axisymmetric flow in polar coordinate system is

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial s}{\partial r} - \frac{s}{L^2} = \frac{S}{T} \cdot \frac{\partial s}{\partial t} \qquad \ldots(4)$$

where,

s = drawdown in the well,

L^2 = $T/(K'/m')$ and is termed as leakage factor,

K' = permeability of the aquitard, and

m' = thickness of the aquitard.

The leaky aquifer is diagrammatically represented in Figure1 While deriving the equations the following assumptions are made; Viz.,

1) The aquifer and aquitard are individually elastic, homogeneous, isotropic, uniform in thickness, infinite in aerial extent and the tangent of the angle of dip is small.
2) The conductivity and specific storage remain constant with time and constant in the space of the layer they characterize; and
3) The wells are screened through the aquifer portion only.

Though, he derived an exact equation for drawdowns in the main aquifer, the inverse Laplace transform present in the equation could not be evaluated. Hence he could present only asymptotic solutions at later and early times.

Neuman and Witherspoon (1969 a and b) later extended Hantush's work to obtain a complete solution to a more complex problem of flow in a confined system consisting of two aquifers separated by an aquitard. They also developed asymptotic solutions for small values of time. All the available analytical solutions are based on the assumption that the flow direction is horizontal in the aquifer and vertical in the aquitard. Hantush also assumed that the leakage from the aquitard into the aquifer is vertical and the storativity of the aquitard is neglected. This assumption was shown by Neuman and Witherspoon (1969) to lead to errors of no more than 5% when the permeability of the aquifer is at least two orders of magnitude greater than that of the aquitard. This was later substantiated by Javandel and Witherspoon (1969) while they were analyzing the transient fluid flow through multilayered aquifers.

2.5 Flow towards partially screened well without storage in leaky aquifers

Wells for which the water entry section is less than the thickness of the aquifer they penetrate, are called as partially screened wells (Figure 2). Unlike the flow towards a completely screened well where the main flow takes place essentially in planes parallel to the bedding planes of the aquifer the flow towards partially screened wells is three dimensional. Consequently the drawdown observed in partially screened wells will depend among other variables, on the length and space position of the screened portion (i.e., water entry section) of the discharging well. Convergence of flow at a well of partial screening causes additional head losses, which are superimposed on those of an undeformed radial flow pattern of fully penetrating well. Solutions developed by many authors are obtained therefore in terms of the two drawdown components; the first being caused by the well of complete penetration, while the second indicates the effect of partial screening. However, the solutions differ somewhat mainly by the assumed distribution of pressure along the well surface. Steady state analysis of partially screened wells in confined aquifers was

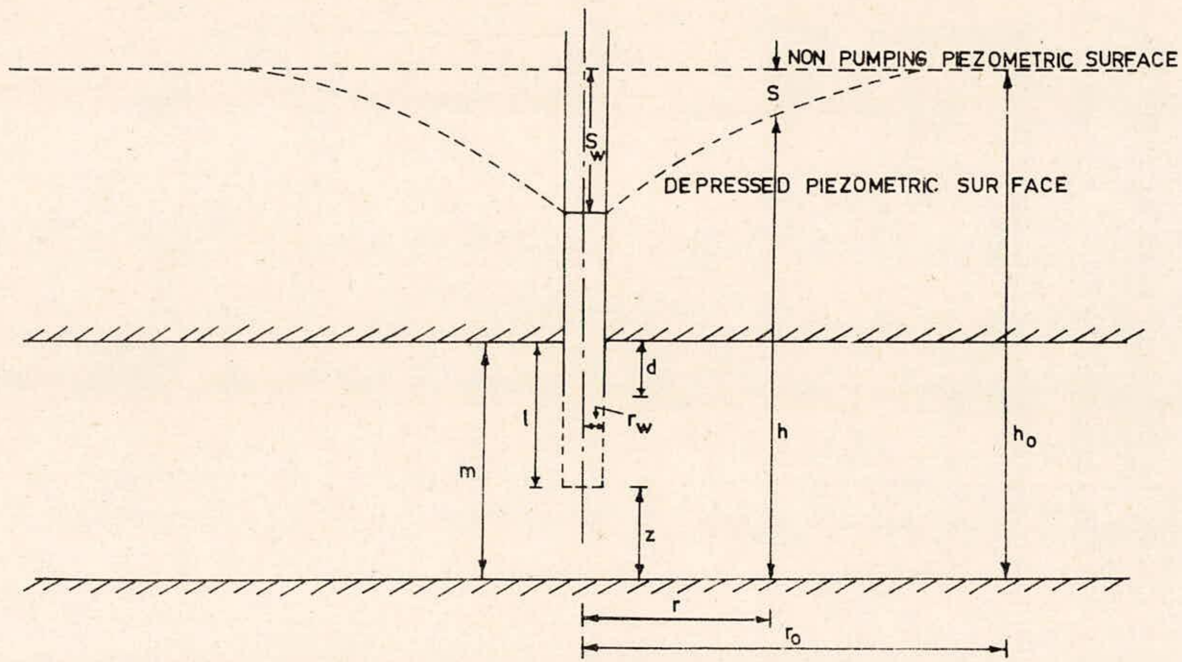

FIG. 2 - DIAGRAMMATIC SKETCH OF PARTIALLY PENETRATING AND PARTIALLY SCREENED WELL IN A CONFINED AQUIFER

under taken by Forchheimer (1930),De Glee (1930), Muskat (1932) Li,et al,(1954), Kirkham(1959) and others. Hantush(1961) gave early and late time solutions for the transient flow towards partially screened wells in confined aquifers. The partially screened wells in thick leaky artesian aquifers were analyzed by Hantush (1967) and Halepaska(1972). For this boundary value problem, Hantush gave the solution for drawdown involving sine and cosine series.

3.0 PROBLEM DEFINITION AND FORMULATION

The analysis of wells with storage in confined aquifers revealed that the use of Theis curve for the estimation of aquifer parameters lead to erroneous results during early time pumping history. From the analysis, it was found that the effect of storage of the well would be negligible at late times or after certain distance from the discharging well. The storage effected zone both in spatial and time coordinates was determined beyond which the applicability of Theis curve can be made for determining the aquifer constants. Also, type curves for the wells with storage in confined aquifers were presented which can be used for assessing transmissivity and storage coefficient from the early pumping history.

The present investigation is primarily concerned with study and anslysis of the effects of storage of a partially screened pumping well on the flow field in leaky(semi confined) aquifer under constant discharge conditions. As a special case of the above mentioned problem the analysis include fully penetrating wells also.

3.1 Description of the flow field

A diagrammatic sketch of a partially screened well in a semi-confined aquifer (i.e., an aquifer overlain and/or underlain by an aquitard) is shown in Figure 3. The aquifer-aquitard system is considered to be of infinite aerial extent with the well at its centre such that all physical conditions are

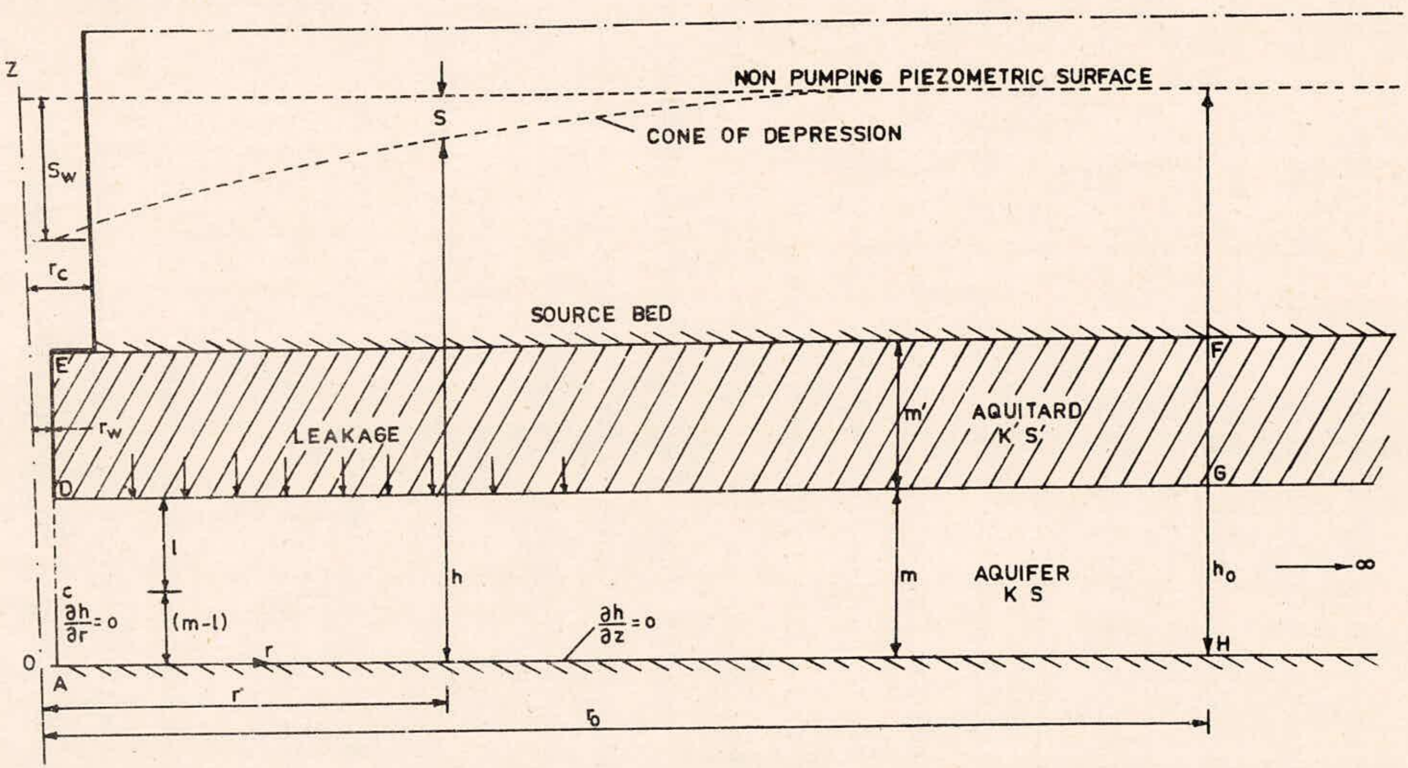

FIG. 3-PARTIALLY PENETRATING WELL WITH STORAGE IN A SEMI CONFINED AQUIFER

symmetrical with respect to the axis of the well. The aquifer of thickness, m , is overlain by an aquitard of uniform thickness, m', and underlain by an aquiclude. Both the aquifer and aquitard are independently considered to be homogeneous and isotropic and they have negligible dip. The well of radius, r_w, has penetrated completely the aquifer and aquitard. However, it is screened only through a length l, in the aquifer region commencing from the junction of the aquifer and aquitard. The well is considered to be pumped at constant discharge Q, and the steady state conditions are yet to reach. The casing of the well, r_c, starts from the top of the aquitard. The drawdown in the well at any time t (total time from the starting of the pump) is designated by s_w. The height of the non-pumping piezometric surface is indicated by h_o. For the purpose of discretization, the infinite aquifer-aquitard system is replaced by a finite system, with the inflow potential boundary for the aquifer located at $r = r_o$ from the axis. The height of the piezometric surface at any radial distance r is designated by h. The water table surface is assumed to be at a constant height H, over the surface of the aquitard.

3.2 Mathematical Formulation

3.2.1 Assumptions

The following simplifying assumptions have been made in the study of the problem -

1) the aquifer as well as the aquitard are individually homogeneous and isotropic with a negligible dip.

2) Darcy's law is assumed over the entire domain.

3) Water is released instantaneously from the aquifer with decline in head.

4) The water table above the aquitard is remaining at a constant level.

5) The model is considered as a composite unit of aquifer-aquitard system.

3.2.2 Governing differential equation

The governing differential equation is same as the equation 2 since the aquifer and aquitard is treated as a composite unit.

3.2.3 Boundary conditions

The boundary conditions for this problem are as follows (Figure 3).

AC	$r = r_w$	$0 \leq z \leq (m - 1)$	$\delta h/\delta r = 0$	...(5)
DE	$r = r_w$	$m \leq z \leq (m + m')$	$\delta h/\delta r = 0$	...(6)
EF	$z = m+m'$	$r_w \leq r \leq r_o$	$h = H$	...(7)
FG	$r = r_0$	$m \leq z \leq (m+m')$	$h = H$	...(8)
GH	$r = r_0$	$0 \leq z \leq m$	$h = h_o$	...(9)
HA	$z = 0$	$r_w \leq r \leq r_0$	$\delta h/\delta z = 0$	...(10)
CD	$r = r_w$	$(m-1) \leq z \leq m$	the discharge prescribed condition	

$$\text{i.e.} \; -2\pi K \int_{z=(m-1)}^{m} r_w \left[\frac{\partial h(t)}{\partial r}\right]_{r=r_w} dz + \pi r_c^2 \frac{\partial}{\partial t} h_w(t) = -Q \quad \text{...(11)}$$

For the case of fully screened well, the equations 5 and 10 are replaced by the following equation :

AD $\quad r = r_w \quad 0 \leq z \leq m \quad$ the discharge prescribed condition

$$\text{i.e.} -2\pi K \int_{z=0}^{m} r_w \left[\frac{\partial h(t)}{\partial r}\right]_{r=r_w} dz + \pi r_c^2 \frac{\partial h_w(t)}{\partial t} = -Q \qquad \ldots(12)$$

It may be noted that $(\partial h/\partial r)_{r=r_w}$ varies with z for a partially screened well, while it is a constant for a fully screened well.

3.2.4 Initial conditions

The initial condition is

$$h(r,o) = h_o \qquad \ldots(13)$$

The boundary and initial conditions as indicated by equations 6 to 13 combined with the governing differential equation (equation 2) will define the well with storage in a leaky aquifer. The solution of equation 2 with initial and boundary coordination will give head variation across the aquifer.

4.0 FINITE ELEMENT FORMULATION

By employing the variational principle in conjunction with the finite element idealization, a powerful solution technique is available for the determination of unknown function distribution within complex bodies of arbitrary geometry. Zienkiewicz, et al., (1966) have employed the finite element method in obtaining steady state solutions to hetereogenous and anisotropic seepage problems. Finn(1967), Taylor and Brown (1967), Neuman and Witherspoon (1969) have used this method to investigate steady state flow involving a free surface. Complex transient problems have been studied by Javandel and Witherspoon (1968,1969) and Witherspoon et al., (1968) using the finite element method. Sandhu and Wilson (1969) made a finite element analysis of soil consolidation (i.e. seepage in an elastic medium). Fenton (1968) and Volker (1969) used this method to investigate steady non Darcy flow with a free surface. Mc Corquidale (1970) employed this method to study both steady and transient non Darcy flow with a free surface. Also, France, et al.,(1972) and Verruijt (1972) have applied this method in solving unsteady flow problems. Chowdary and King (1972) discussed the analysis of non Darcy steady seepage problems using the finite element techniques. This method is also adopted by Huyakorn (1974) while studying the two regime flow around wells in confined and unconfined aquifers. Das (1975), Rao and Das (1976) have applied this method to analyse the flow pattern around the partially penetrating wells in unconfined aquifers. Morandi and Mancino (1976) used the finite element method to simulate the confined flow. Seethapathi (1979) used the finite element technique to study the problems of wells with storage.

4.1 Development of Variational Principle

Variational forms of the previously derived equations may be obtained by considering an equivalent variational problem and adopting Euler-Lagrange equation from the Calculus of variations. The general aquifer system is shown in Figure 4 in Cartesian coordinate system, with x_1, x_2, x_3 as coordinates.

Let h (x_1, t) be an admissible function with the second order space and first order time derivatives, which are continuous everywhere in the given flow region R and let the time domain be sub-divided into a number of finite time increments.

The general functional to be minimised over the space region R over the time increment Δt may be expressed as

$$\left[I\ (h)\right]_R = \int_t^{t+\Delta t} \int_R F\left(h, \frac{\partial h}{\partial x_i}, \frac{\partial h}{\partial t}, x_i, t\right)\, dR\, dt \qquad \ldots(14)$$

where i refers to a particular component along the coordinate axis and its range is one to three for three dimensional space region.

Now the problem is reduced to seeking the function h(x,t) which holds the above functional stationary. A necessary condition is the Euler Lagrange equation [Myers (1959), Remson, Hornberger and Moltz (1971) and Huyakorn (1974)] which can be written as

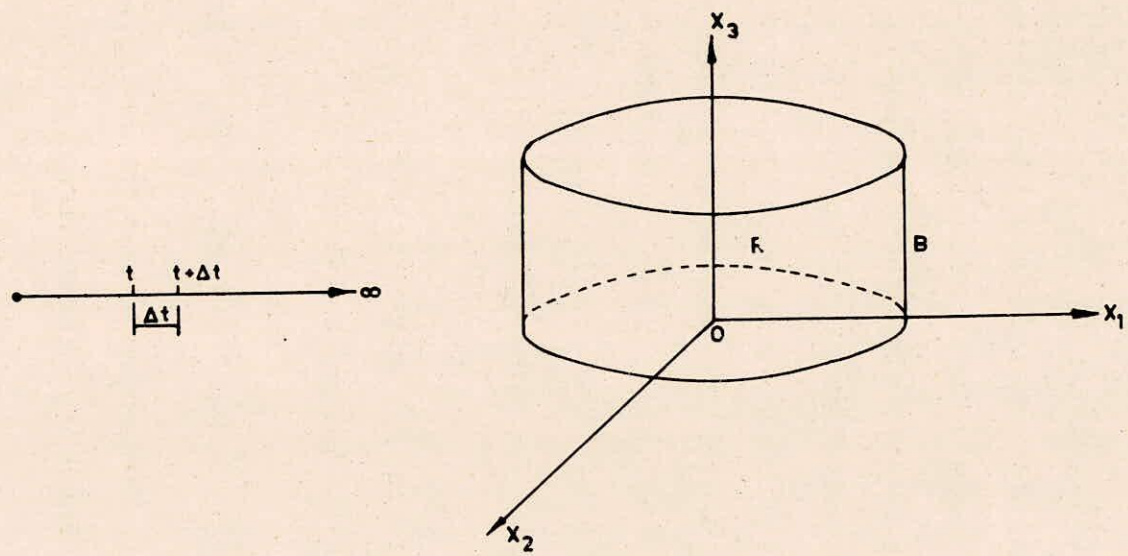

FIG. 4 - GENERAL AQUIFER SYSTEM IN CARTESIAN COORDINATES

$$\frac{\partial F}{\partial h} - \frac{\partial}{\partial x_i} \cdot \frac{\partial F}{\left(\frac{\partial h}{\partial x_i}\right)} + \frac{\partial}{\partial t} \cdot \frac{\partial F}{\partial\left(\frac{\partial h}{\partial t}\right)} = 0 \qquad \ldots(15)$$

The above equation represents various classes of partial differential equations. The equation derived earlier (equation 1) belong to one of these classes.

To obtain the expression for the functional F, the equations derived earlier may be equated to Equation 15.

The field equation describing Darcy flow through isotropic aquifers is rewritten as

$$- \frac{\partial}{\partial x_1}\left(K \frac{\partial h}{\partial x_i} \right) + S_s \frac{\partial h}{\partial t} = 0 \qquad \ldots(16)$$

On comparing Equation 16 to Equation 15, the following equations are obtained

$$\frac{\partial F}{\partial h} = 0 \qquad \ldots(17)$$

$$\frac{\partial F}{\partial\left(\frac{\partial h}{\partial x_i}\right)} = K \frac{\partial h}{\partial x_i} \qquad \ldots(18)$$

$$\frac{\partial F}{\delta\left(\frac{\partial h}{\partial t}\right)} = h\, S_s \qquad \ldots(19)$$

Integrating the above equations

$$F = 1/2\, K \frac{\partial h}{\partial x_i} \cdot \frac{\partial h}{\partial x_i} + h\, S_s \frac{\partial h}{\partial t} \qquad \ldots(20)$$

Hence the functional over the region R is given by

$$\left[I(h)\right]_R = \int_t^{t+\Delta t} \int_R 1/2\, K \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_i} + h\, S_s \frac{\partial h}{\partial t}) dR\, dt \qquad \ldots(21)$$

4.1.2 Initial conditions

At a particular time, (taken as the initial time) the head distribution throughout the space region of the flow system is assumed to be known. If, in the minimisation of the functional the time integration is carried out between t=0 and t = Δ t the admissible function will automatically satisfy the initial head condition represented by the equation

$$h\ (x_i, 0) = h_o(x_i),\ (x_i) \varepsilon\ \bar{R} \qquad \ldots(22)$$

where $h_o(x_i)$ is the initial non pumping piezometric height and $\bar{R}$ denotes the closed region of the flow system.

4.1.3 Boundary conditions

In minimising the functional, the requirement on the flow boundary must also be met. These requirements lead to the addition of some extra terms to the functional.

For the various types of boundary conditions like head prescribed boundaries, flux prescribed boundaries, impervious boundaries, seepage faces, free surface boundaries etc., the additional terms have been obtained by Neuman and Witherspoon (1971).

On the portion B_1, where flux is prescribed, the additional term may be written as

$$\left[I(h)\right]_{B_1} = \int_t^{t+\Delta t} \int_{B_1} h\ \bar{q}\ dB\ dt \qquad \ldots (23)$$

On the portion B_2 where the function h is prescribed, it is given by

$$\left[I(h)\right]_{B_2} = \int_t^{t+\Delta t} \int_{B_2} (h-\bar{h})\ v_i \cdot n_i\ dB\ dt \qquad \ldots(24)$$

where $\bar{h}$ is the prescribed head on the boundary and v is the velocity component.

Since the admissible function h is chosen to automatically satisfy the prescribed head condition on the entire flow boundary, the term contributed by B_2 may be dropped. For the impervious boundaries, the normal component of the velocity across the boundary is zero and hence $v_i n_i = 0$ and there will be no contribution to the functional from this type of boundary condition.

Now the functional for the entire region $\bar{R}$ with the above mentioned boundary conditions being incorporated, becomes

$$\left[I\ (h)\right]_{\bar{R}} = \left[I\ (h)\right]_{R} + \left[I\ (h)\right]_{B_1} \qquad \ldots(25)$$

4.2 Formulation of element matrices

Consider the general problem of three dimensional transient flow towards a pumping well penetrating a leaky aquifer. The flow region $\bar{R}$ consists of the region inside the boundary and the flow boundary. The functional over $\bar{R}$, may be expressed as the sum of the functional over the interior Region R, and over the boundary B (Equation 25).

The flow region is discretised into a network consisting of M interconnected finite elements.

Considering a typical element with a closed sub-region of $\bar{R}^e$ and with M^e nodes on its boundary, the head distribution within the element may be written as

$$h\ (x_i,t) = N_I(x_i)\ h_I(t) \qquad \ldots(26)$$

where $N_I(x_i)$ are piecewisely defined functions of coordinates

(x_1, x_2, x_3) within the element, $h_I(t)$ are the nodal values at time t of the function 'h', and the repeated subscript I denotes summation over the full range, from 1 to M^e. The functional over the entire flow region $\left[I\ (h)\right]_{\bar{R}}$, may now be expressed as the sum of the functionals over the elements, thus

$$\left[I\ (h)\right]_{\bar{R}} = \sum_{1}^{M} \left[I^e\ (h)\right] \qquad \ldots(27)$$

For convenience, the elements are classified into two categories viz., interior elements, which have their closed elemental boundaries contained within the interior of the flow region, and exterior elements, which have their elemental boundaries as parts of the boundary of the flow region. Accordingly, the elemental contributions are evaluated.

4.2.1 Interior elements

For the interior elements, the functional R^e is given by

$$I^e(h) = \int_{t}^{t+\Delta t} \int_{R^e} \left(\tfrac{1}{2} K \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} + S_s\ h \frac{\partial h}{\partial t}\right) dR\ dt \qquad \ldots(28)$$

Differentiating the above equation with respect to h_I, gives

$$\frac{\partial I^e}{\partial h_I} = \int_{t}^{t+\Delta t} \int_{R^e} \left[K \frac{\partial h}{\partial x_j} \frac{\partial}{\partial h_I}\left(\frac{\partial h}{\partial x_i}\right) + S_s\ h \frac{\partial}{\partial h_I}\left(\frac{\partial h}{\partial t}\right) + S_s \frac{\partial h}{\partial t} \frac{\partial h}{\partial h_I}\right] dR\ dt \qquad \ldots(29)$$

From equation 26, it follows that

$$\frac{\partial h}{\partial x_i} = \frac{\partial N_I}{\partial x_i}\ h_I \qquad \ldots(30)$$

$$\frac{\partial}{\partial h_I}\left(\frac{\partial h}{\partial x_i}\right) = \frac{\partial N_I}{\partial x_i} \qquad (31)$$

Also, since N_I are functions which do not vary with time, it follows that

$$h \frac{\partial}{\partial h_I} \left(\frac{\partial h}{\partial t}\right) = N_J \, h_J \frac{\partial N_I}{\partial t} = 0 \qquad \ldots(32)$$

and

$$\frac{\partial h}{\partial t} \cdot \frac{\partial h}{\partial h_I} = \frac{\partial h_J}{\partial t} \, N_J \, N_I \qquad \ldots(33)$$

substituting equations 31,32 and 33 into equation 29 gives

$$\frac{\partial I^e}{\partial h_I} = \int_t^{t+\Delta t} \int_{R^e} \left[K \frac{\partial N_J}{\partial x_j} \, h_J \frac{\partial N_I}{\partial x_i} + S_s \frac{\partial h_J}{\partial t} \, N_J N_I \right] dR \, dt \qquad \ldots(34)$$

4.2.2 Exterior elements

For the evaluation of the functional contributed by an exterior element, the boundary conditions on the element boundary has to be taken into account, for which some extra terms are being added for the functional that is already derived for an interior element. These terms only exist on the exterior portion of the element boundary and vanish elsewhere. In the case of leaky flow problems as attempted in this investigation the boundary conditions are prescribed head and prescribed flux conditions. If B_1^e and B_2^e denote the exterior portions of an element where the flux and head functions are described respectively, the additional terms are given by

$$\int_t^{t+\Delta t} \int_{B_1^e} h \, \bar{q} \, dB \, dt \quad \text{and} \quad \int_t^{t+\Delta t} \int_{B_2^e} (h-\bar{h}) \, v_i n_i \, dB \, dt$$

The admissible function h is chosen to automatically satisfy the prescribed head condition on the entire flow boundary. Therefore, the contribution of the term due to head prescribed boundary, may be dropped.

Now the functional is

$$[I(h)]_{\bar{R}^e} = \int_t^{t+\Delta t} \int_{R^e} \left(\tfrac{1}{2} K \frac{\partial h}{\partial x_j} \frac{\partial h}{\partial x_i} + S_s \frac{\partial h}{\partial t} \right) dR\, dt \quad +$$

$$\int_t^{t+\Delta t} \int_{B_1^e} h\, \bar{q}\, dB\, dt \qquad \ldots(35)$$

Differentiating the equation 35 with respect to h_I, gives

$$\frac{\partial}{\partial h_I}(I^e) = \int_t^{t+\Delta t} \int_{R^e} \left(K \frac{\partial N_J}{\partial x_j} h_J \frac{\partial N_I}{\partial x_i} + S_s \frac{\partial h_J}{\partial t} N_J N_I\right) dR\, dt$$

$$+ \int_t^{t+\Delta t} \int_{B_1^e} \bar{q}\, N_I\, dB\, dt \qquad \ldots(36)$$

For convenience the following terms have been introduced

$$COD^e_{JI} = \int_{R^e} K \frac{\partial N_J}{\partial x_j} \frac{\partial N_I}{\partial x_i} dR \qquad \ldots.(37)$$

$$POR^e_{JI} = \int_{R^e} S_s \; N_J \; N_I \; dR \qquad \ldots(38)$$

$$DIS^e_I = \int_{B_1^e} \bar{q}\; N_I \; dB \qquad \ldots(39)$$

Substituting the above equations 37 to 39 in 34 and 36 the following expressions are obtained for interior and exterior elements respectively in the case of Darcy flow.

$$\frac{\partial I^e}{\partial h_I}(\text{for interior}) = \int_t^{t+\Delta t} COD^e_{JI}\, h_J\, dt + \int_t^{t+\Delta t} POR^e_{JI} \frac{\partial h_J}{\partial t}\, dt \qquad \ldots(40)$$

and

$$\frac{\partial I^e}{\partial h_I}(\text{for exterior}) = \int_t^{t+\Delta t} COD^e_{JI}\, h_J\, dt$$

$$+ \int_t^{t+\Delta t} POR^e_{JI} \frac{\partial h_J}{\partial t}\, dt + \int_t^{t+\Delta t} DIS^e_I\, dt \qquad \ldots(41)$$

The following matrix notation is used for convenience.

$$[COD]^e = \int_{R^e} K [S]^T [S] \, dR \qquad \ldots (42)$$

$$[POR]^e = \int_{R^e} S_s [N]^T [N] \, dR \qquad \ldots (43)$$

$$[DIS]^e = \int_{B_1^e} \bar{q} \, [N]^T dB \qquad \ldots (44)$$

where $[N]$ is tne shape function matrix given by

$$[N] = [N_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots N_{M^e}] \qquad \ldots (45)$$

$[N]^T$ = Transpose of N

$[S]^T$ = Transpose of matrix $[S]$ given by

$$[S] = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \ldots\ldots\ldots\ldots & \frac{\partial N_{M^e}}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & \ldots\ldots\ldots\ldots & \frac{\partial N_{M^e}}{\partial x_2} \\ \frac{\partial N_1}{\partial x_3} & \ldots\ldots\ldots\ldots & \frac{\partial N_{M^e}}{\partial x_3} \end{bmatrix}$$

The equations relating the differentials of the functional $I^e(h)$ and the nodal values of the function h may also be written in the matrix form. Rewriting the equations 40 and 41 in the matrix form we obtain

$$\left[\frac{\partial}{\partial h_I}(I^e)_{\text{interior}}\right] = \int_t^{t+\Delta t} [COD]^e \, [h^e] \, dt + \int_t^{t+\Delta t} [POR]^e \left[\frac{\partial h}{\partial t}\right]^e dt \qquad \ldots (46)$$

$$\left[\frac{\partial}{\partial h_I}(I^e)_{\text{exterior}}\right] = \int_t^{t+\Delta t} [COD]^e \, [h^e] \, dt + \int_t^{t+\Delta t} [POR]^e \left[\frac{\partial h}{\partial t}\right]^e dt + \int_t^{t+\Delta t} [DIS]^e \, dt \qquad \ldots (47)$$

$$\left[\frac{\partial I^e}{\partial h}\right] = \begin{bmatrix} \frac{\partial I^e}{\partial h_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial I^e}{\partial h_{M^e}} \end{bmatrix} \quad \text{and} \quad \left[h^e\right] = \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ \cdot \\ h_{M^e} \end{bmatrix}$$

Since the present study is concerned with flow towards wells, axisymmetric system is adopted. The domain is divided into a finite number of ring shaped elements, with rectangular cross section (Figure 5). A triangular element with linear variations of head has been chosen in the present analysis. To save the requirements of computer storage, the following technique has been used. The aquifer is discritized as an assemblage of rectangular elements. By drawing the diagonals, each rectangle is subdivided into four triangles with five nodes. The overall conductance and porosity matrices have been obtained by combining the corresponding individual matrices of the triangles. These 5 x 5 matrices have been condensed to 4 x 4 matrices by algebriac elimination of the central node.

4.3 Assemblage of elements

In the element assembling process, all elements are assembled through the specification of the reduced compatibility condition, which requires that the nodal values of the function be the same at the coincident nodes of adjacent elements and also equal to prescribed value on the boundary portion where the function is prescribed.

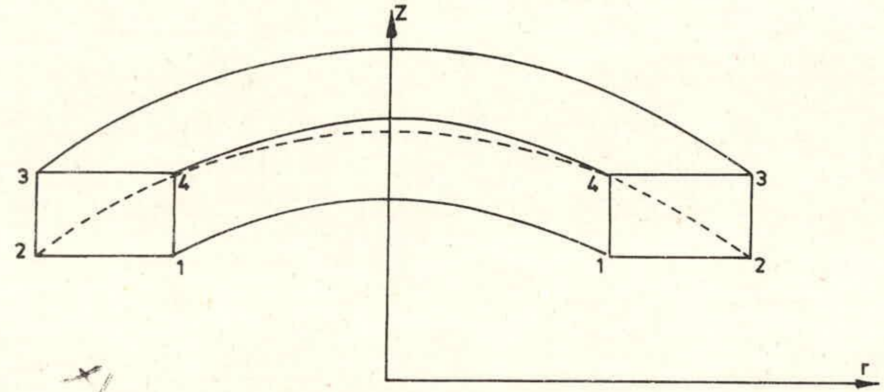

FIG. a

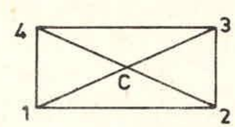

FIG. b

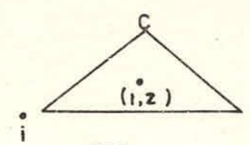

FIG. c

FIG. 5 - RECTANGULAR ELEMENT IN AXISYMMETRIC FLOW

Thus an assembling, the functional for the entire flow region becomes

$$\left[I\ (h_I)\right]_{\overline{R}} = \sum_{e=1}^{M} I^e\ (h_I) \qquad \text{for } I = 1 \ldots\ldots M \qquad \ldots(48)$$

where the summation is taken over the elements adjacent to the I-th nodal point and the subscript I ranges from one to the total number of nodes in the entire flow region. The minimization of $\left[I\ (h_I)\right]_{\overline{R}}$ requires that

$$\frac{\partial}{\partial h_I}\left[I\ (h_I)\right]_{\overline{R}} = \sum_e \frac{\partial I^e}{\partial h_I} = 0 \qquad \ldots(49)$$

$$\text{for } I = 1 \ldots\ldots M$$

Thus the general expression is now written as

$$\frac{\partial I^e}{\partial h_I} = \int_t^{t+\Delta t} COD^e_{JI}\ h_I\ dt + \int_t^{t+\Delta t} POR^e_{JI} \frac{\partial h_J}{\partial t}\ dt$$

$$+ \int_t^{t+\Delta t} DIS^e_I\ dt \qquad \ldots(50)$$

substitution of equation 50 into equation 49 leads to

$$\sum_e \int_t^{t+\Delta t} COD^e_{JI}\ h_J\ dt + \sum_e \int_t^{t+\Delta t} POR^e_{JI}\ \frac{\partial h_J}{\partial t}\ dt + \sum_e \int_t^{t+\Delta t} DIS^e_I\ dt = 0$$

$$\ldots(51)$$

Introducing the following gross matrices

$$C_{JI} = \sum_e COD^e_{JI} \qquad \ldots(52)$$

$$P_{JI} = \sum_e POR^e_{JI} \qquad \ldots(53)$$

$$Q_I = \sum_e DIS^e_I \qquad \ldots(54)$$

where C is called conductance matrix, P is called porosity matrix and Q is the prescribed boundary matrix

Substitution of the equations 52 to 54 into equation 51 results in

$$\int_t^{t+\Delta t} C_{JI}\, h_J\, dt + \int_t^{t+\Delta t} P_{JI}\, \dot{h}_J\, dt + \int_t^{t+\Delta t} Q_I\, dt = 0 \quad \ldots(55)$$

where subscripts J and I range from one to the total number of nodes in the entire region. Rewriting the above equation we have

$$C\ h + P\ \dot{h} + Q = 0 \text{ at any instant of time} \quad \ldots(56)$$

where $\dot{h}$ indicates the derivative of head w.r.t. time.

4.4 Integration with respect to time

Equation 55 represents a system of M simultaneous equations involving the integral terms which must be integrated with respect to time. To carry out the integration, it is assumed that all the nodal values of h_I and Q_I are known at earlier time t. Using the Crank Nicolson method with a logarithmic time increment, the final equations would be,

$$[D]\{x\} = [E]\ \{h\} - \frac{\Delta t}{2}\ (\{Q\} + \{Q_M\}) \quad \ldots.(57)$$

where,

$$[D] = [P] + \frac{\Delta t}{2}[C] \quad \ldots..(58)$$

$$[E] = [P] - \frac{\Delta t}{2}[C] \quad \ldots..(59)$$

$\{x\}$ = nodal head vector at time, t+1

$\{h\}$ = nodal head vector at time,t

$\{Q\}$ = boundary discharge vector at time,t,

$\{Q_M\}$ = boundary discharge Vector at time, t+1

Δt = time increment between t and t+1

4.5 Implementation of boundary conditions

4.5.1 Head prescribed condition

While minimizing the functional for the elements and while assembling them to generate the conductance and porosity matrices, it was assumed that even at the nodes where head is prescribed, that nodal values were unknown and differentiation with respect to that nodal value was performed, for the sake of convenience and ease. However, in reality, the head at that node under consideration would be a constant value and eaual to the prescribed value. To implement this condition, without altering the structure of the gross matrices, a suitable procedure was adopted.

4.5.2 Treatment of conditions on the Well Boundary

Two types of well boundary conditions are possible, depending on the pumping operation. If the well is pumped at a constant discharge, the constant prescribed flow rate will prevail. On the other hand, it is pumped such that the water level in the well remains constant, the constant prescribed head condition will result. These two types of boundary conditions were dealt with in the following manner.

(a) Constant head prescribed boundary condition

Consider a typical pumped well as shown in the Figure 6 As indicated in the figure, if the water level is maintained constant throughout the pumping period, the head values at the

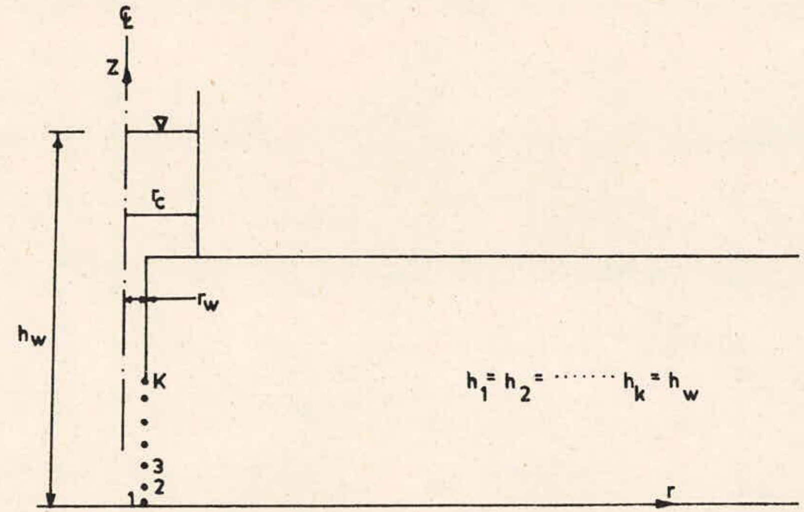

FIG. 6 - THE BOUNDARY OF A TYPICAL WELL

nodes situated on the well screen will be constant with time and equal to the known elevation of the water level above the datum plane. Let (a, b) and (i, j) be additional subscripts referring to the nodes situated on the well screen where the boundary condition is prescribed and the remaining nodes in the flow region respectively. It follows that the subscripts a and b ranges from 1 to K and i, j ranges from K + 1 to M. The equation 57 is partitioned into two parts (1) upto the rows equal to the number of nodes on the well boundary where the boundary condition is prescribed, (ii) remaining nodes. Thus

$$\begin{bmatrix} D_{ab} & D_{aj} \\ \hline D_{ib} & D_{ij} \end{bmatrix} \begin{Bmatrix} x_b \\ - \\ x_j \end{Bmatrix} = \begin{bmatrix} E_{ab} & E_{aj} \\ \hline E_{ib} & E_{ij} \end{bmatrix} \begin{Bmatrix} h_b \\ - \\ h_j \end{Bmatrix}$$

$$- \frac{\Delta t}{2} \begin{Bmatrix} Q_b & - & QM_b \\ \hline Q_j & & QM_j \end{Bmatrix} \qquad \ldots (60)$$

Rewriting the partitioned matrix equation, we have

$$[D_{a\,b}]\{x_b\} + [D_{a\,j}]\{x_j\} = [E_{a\,b}]\{h_b\} + [E_{a\,j}]\{h_j\} - \frac{\Delta t}{2}\{Q_b + QM_b\} \qquad \ldots (61)$$

and

$$[D_{i\,b}]\{x_b\} + [D_{i\,j}]\{x_j\} = [E_{i\,b}]\{h_b\} + [E_{i\,j}]\{h_j\} - \frac{\Delta t}{2}\{Q_j + QM_j\} \qquad \ldots (62)$$

for the boundary condition prevailing on the well face, x_b which signifies the head values at nodal points on the well screen is a known priori. Hence, $x_b = h_b$. Rearranging the terms in the equation 62.

$$[D_{ij}]\{x_j\} = [E_{ib} - D_{ib}]\{h_b\} + [E_{ij}]\{h_j\} - \frac{\Delta t}{2}\{Q_j + QM_j\} \qquad \ldots(63)$$

In the above equation Q_j and QM_j indicate the flux at time t and at time $t+\Delta t$ at the internal nodes, which is equal to zero, since the flux at the internal nodes balances out. Also, substituting the expressions for matrix [D] and [E] (Equations 58 and 59), we have

$$[D_{ij}]\{x_j\} = -\Delta t\,[C_{ib}]\{h_b\} + [E_{ij}]\{h_j\} \qquad \ldots(64)$$

In the equation 64 the right hand side of the equation can be evaluated. Using the Gaussian elimination technique, the equation is solved for x_j . To calculate the flux at the nodal points on the well screen, equation 61 can be used in the form

$$\{QM_b\} = -2\,([C_{ab}]\{h_b\}) + 2\,([E_{aj}]\{h_j\} / \Delta t) - -2\,([D_{aj}]\{x_j\} / \Delta t) - \{Q_b\}) \qquad \ldots(65)$$

However, the discharge from the well would be equal to the sum of the discharge from the aquifer which is the sum of the nodal discharges on the well screen obtained by solving the equation 65 and the discharge obtained from the storage in the well

i.e. $Q_{well} = Q_{aquifer} + Q_{well\ storage}$ ...(66)

(b) Constant prescribed flow rate condition

In the extraction of groundwater pumping, it is common to maintain constant total discharge from the well throughout the pumping period. Accordingly, since the total flow rate is fixed, the water level in the well and prescribed hydraulic head along the screened portion of the well boundary must vary with time. Once again consider the well shown in Figure 6 . If $\bar{Q}$ denotes the prescribed flow rate, the condition is given by

$$\bar{Q} = Q_{aquifer} + Q_{well\ storage}$$

$$Q_{aquifer} = \sum_{b=1}^{K} Q_b \quad \text{and} \qquad \ldots(67)$$

$$Q_{well\ storage} = \pi r_c^2 \, (s_{t+1} - s_t) \qquad \ldots(68)$$

where s = is the drawdown in the well at any instant of time.

In the simple case where the well storage is ignored and where the total discharge is uniformly distributed along the well screen as in a fully screened well, the constant flow rate condition may be treated by computing the values Q_b from equation 67 and incorporating this into equation 61.

However, in the case of partially screened well, the discharge along the well bore is not uniformly distributed and further it is not known a priori. To implement this boundary condition in the computer programme an iterative technique has been employed. This procedure is essentially the same as outlined for problems with head prescribed boundary conditions. A guess is made at any time 't' and is corrected iteratively till the discharge boundary condition is satisfied in the limits of the accuracy prescribed. The details are described below.

Let t_{n+1} and t_n denote the current and preceeding times respectively. For the first iteration, k=1, an initial estimate of the value of H at time t_{n+1} is made from

$$(H)^1_{t_{n+1}} = (H)_{t_n} + \Delta H \quad \text{for} \quad n = 0,1 \qquad \ldots(69)$$

or from the following logarithmic extrapolcation formula

$$(H)^1_{t_{n+1}} = \frac{\log(t_{n+1}/t_{n-1})}{\log(t_n / t_{n-1})} \quad (H)_{t_n} - (H)_{t_{n-1}} \qquad \ldots(70)$$

when the equation 69 is used an initial guess had to be made to the value of the head increament ΔH at the beginning of the first and second time steps. The initial estimate of H is used in solving for the unknown head and flux values in equations 64 and 65 respectively. Knowing the nodal flux Q_b and the drawdown between the time increment the total discharge Q^k is calculated by the equation

$$Q^k = \sum_{b=1}^{K} Q_b + \frac{\pi r_c^2}{\Delta t} \left[(H)_{t_n} - (H)^1_{t_{n+1}} \right] \qquad \ldots(71)$$

The Q^k, thus obtained, is compared with the prescribed discharge $\bar{Q}$.

$$\text{If} \quad \left| \frac{Q^k - \bar{Q}}{\bar{Q}} \right| > \varepsilon \qquad \ldots(72)$$

where ε is the prescribed tolerance of the discharge ratio, a new trial head is calculated from

$$(H)^2_{t_{n+1}} = (H)_{t_n} + \left[(H)^1_{t_{n+1}} - (H)_{t_n} \right] \frac{\bar{Q}}{Q^1} \qquad \ldots(73)$$

The solution for the unknown nodal head and flux values is then repeated and the total discharge is recalculated and tested for convergence. If the convergence is still not obtained, the following formula is applied to modify the head.

$$(H)^{k+1} = (H)^{k-1} + \left[\frac{(H)^{k} - (H)^{k-1}}{(Q)^{K} - (Q)^{k-1}}\right] \left[\bar{Q} - (Q)^{k-1}\right] \quad \ldots(74)$$

The solution procedure is then repeated and equation 74 is reapplied until convergence is resulted. (The formulae represented by equations 69,70,73,74 are suggested by Huyakorn, 1974). The above iteration procedure gave satisfactory results. For earlier times, the convergence criterion is met after three iterations and for later times, after two iterations the convergence is resulted.

4.6 Elimination Scheme

The assemblage of element matrices, after imposing the conditions prevailing on the well boundary, reduced to a system of (M-K) equations as represented by equation 64 while formulating the gross matrices, the symmetry and banded nature of these matrices is taken into account and hence only upper diagonal elements are calculated and stored, in a rectangular array. The half band width of each matrix is computed as the length between the diagonal element and the last non zero element. This has resulted in eliminating the problem of insufficient computer storage capacity, as only part of the two gross matrices needs to be stored.

A banded Gaussian elimination scheme is employed to solve the reduced system of (M-K) equations. The process of elimination is accomplished by reducing the system of equations to an equivalent triangular form through a series of arithmetic operations on the coefficients of the equations. Then starting from the last equation, the last unknown is solved and the remaining unknowns are obtained by back substitution into the preceeding equations. Since the gross matrices are in a condensed fashion a special subroutine is prepared for this scheme and this requires smaller number of arithmetic operations, which resulted in a considerable saving of the solution time.

5.0 ANALYSIS OF RESULTS

In the present investigation, a mathematical model is developed to study the effects of well storage on the drawdown distribution within the aquifer. The cases of both fully screened and partially screened wells in leaky aquifers have been studied.

5.1 Verification of the Model

Before proceeding with the actual analysis, the programme that has been developed is tested for different standard cases, like, wells in confined aquifers and semi confined aquifers. The results thus obtained were compared with those values presented by Theis and Hantush. It is found that the maximum deviation from these standard values is only of the order of 0.05% indicating that there is a fine agreement between the developed model results and the standard values. The programme is also used to compare the results, for wells with storage in confined aquifers for storage parameter values ranging 10^{-2} to 10^{-5} on the well boundary with those presented by Papadopulous and Cooper (1967). There is a satisfactory agreement of the results obtained by the model developed by author (Table 1).

5.2 Definition of Storage Parameters

While deriving the expression for the temperature distribution for heat flow problems, Carslaw and Jaeger idendified a nondimensional parameter α. The same parameter was subsequently adopted by Papadopulos and Cooper (to define the storage of the large diameter well) to analyse the problems of wells with storage. This parameter reflects the storage properties of the well in comparison to the storage properties of the aquifer. However, the parameter decreases with an increase in the well storage rendering it inconvenient for use in ground water hydrology. Hence, a new parameter called storage parameter is introduced and is designated by β which is reciprocal of α .

5.3 Wells without Storage in Leaky Aquifers

The case of wells without storage in semi confined aquifers is analysed by Hantush. He formulated the problem considering the finiteness of the well. In his anslysis he considered two cases viz., a) the aquitard storage considered b) the aquitard storage neglected. He obtained an expression for the drawdown in terms of inverse Laplace transforms. Since the Inverse transforms are not available, he could not obtain an exact solution. However, asymptotic solutions (at early times and late times) are given by him. While considering the long time solution, he observed that the finiteness of the well has the least significance in time for the expression. At early times, Hantush assumed that the diameter of the well is vanishingly small. Thus it could be seen that Hantush though considered the finiteness of the well in his analysis, he has neglected the same while obtaining the solutions and there is no exact solution for the intermediate times. Hence it can be concluded that exact analytical solutions are not available in literature for wells without storage in leaky aquifers. Hantush assumed that the flow in the aquitard is vertical though it has been subsequently proved by Javendal, Neuman and Witherspoon that the error caused due to such assumption is about 5% when the ratio of the permeabilities of the aquifer to the aquitard is more than 100. Thus the conclusions which can be drawn from Hantush solution from the well storage in leaky aquifers is 1). For a constant discharge from a well, the dradowns will be comparatively smaller than in the case of confined aquifers. For the case of aquitard being more compressible, these drawdowns become still less. 2) At very early times, all the curves corresponding to different value of r/L ratios converge on to the Theis curve. 3) At late times or for the points farther to the well, the drawdowns will attain a particular value and remain constant thereafter.

5.4 Fully screened wells with storage in leaky aquifers

The model that has been developed can be used for a finite diameter well with storage or without storage in an aquifer aquitard system or can also be used for multi-layer aquifers. In the present investigation eight different cases (Figure 7) are used with different values of storage parameters β and for two different aquitard storativity values viz., a) when the storativity of aquitard is equal to the storativity of the aquifer b) when the storativity of the aquitard is 100 times that of the storativity of the aquifer. Finite element solutions are obtained for these cases keeping the aquitard thickness equal

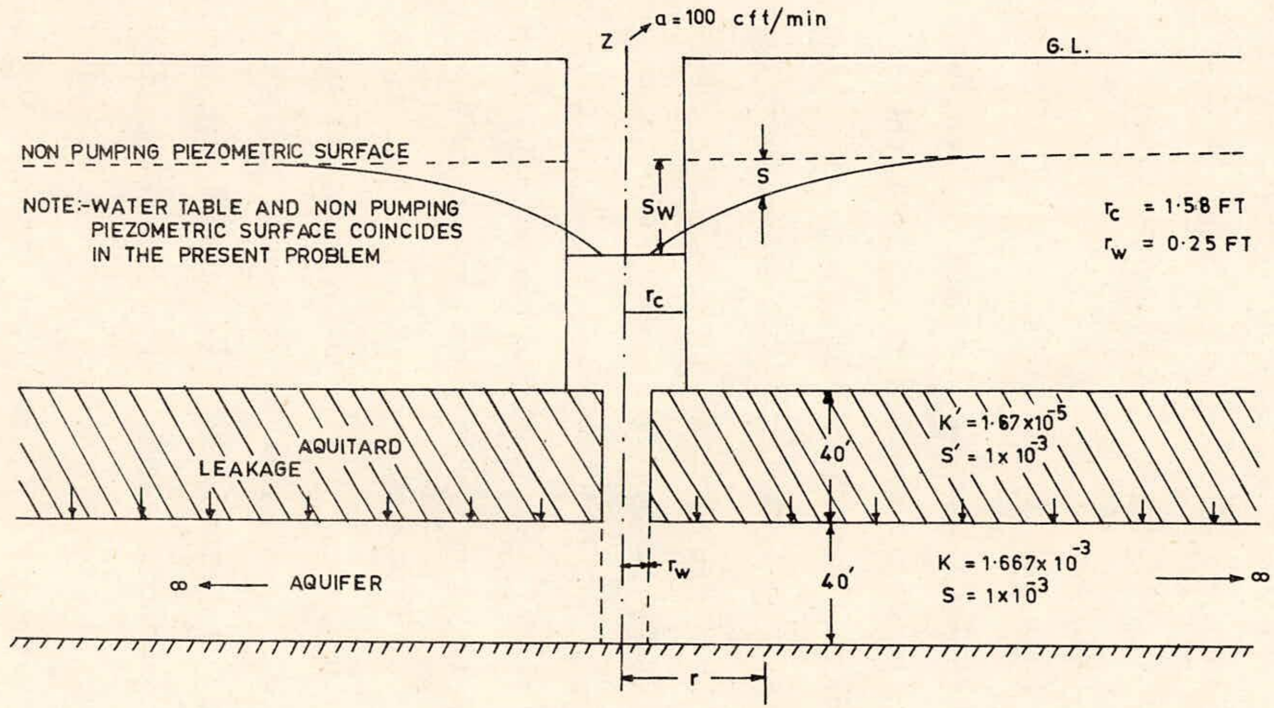

FIG. 7 - SCHEMATIC REPRESENTATION OF THE MODEL.

to that of the aquifer and the permeability ratio between aquitard and aquifer is 1/100. The leakage factor L has been considered as 100 length units. Type curves are drawn between the non-dimensional time parameter (u) with the non-dimensional drawdown (W(u)) on a log-log scale for different values of storage parameters and for different r/L ratios as well as for the two strativity ratios of 1 & 100 respectively.

5.4.1 Shape of the type curves

The shape and general pattern of the type curves for different values of β_s (Figures 8 to 13) for both the cases, S' = S and S' = 100S, are same as the corresponding type curves for the case of non leaky aquifers (confined aquifers). Three regions are seen distinctly in these curves. The straight line portion representing Zone I, is seen to be same as the corresponding region in the non leaky aquifers. However, the Zone II, which represents the combination of aquifer storage and the well storage is found to vary in curvature with r/L ratio . As r/L ratio increases, the curvature in Zone II also increases. In the Zone III, the type curves are similar to those of confined aquifers for small r/L ratios. But, as r/L increases the plots in Zone III become horizontal. The length of the horizontal portion is found to increase with increasing in r/L ratio. Also, it is observed that beyond the r/L ratio of 1.0, the type curves for 'wells with storage' and ' wells with no storage' become almost the same (with an error of about 5% for $\beta = 10^5$)

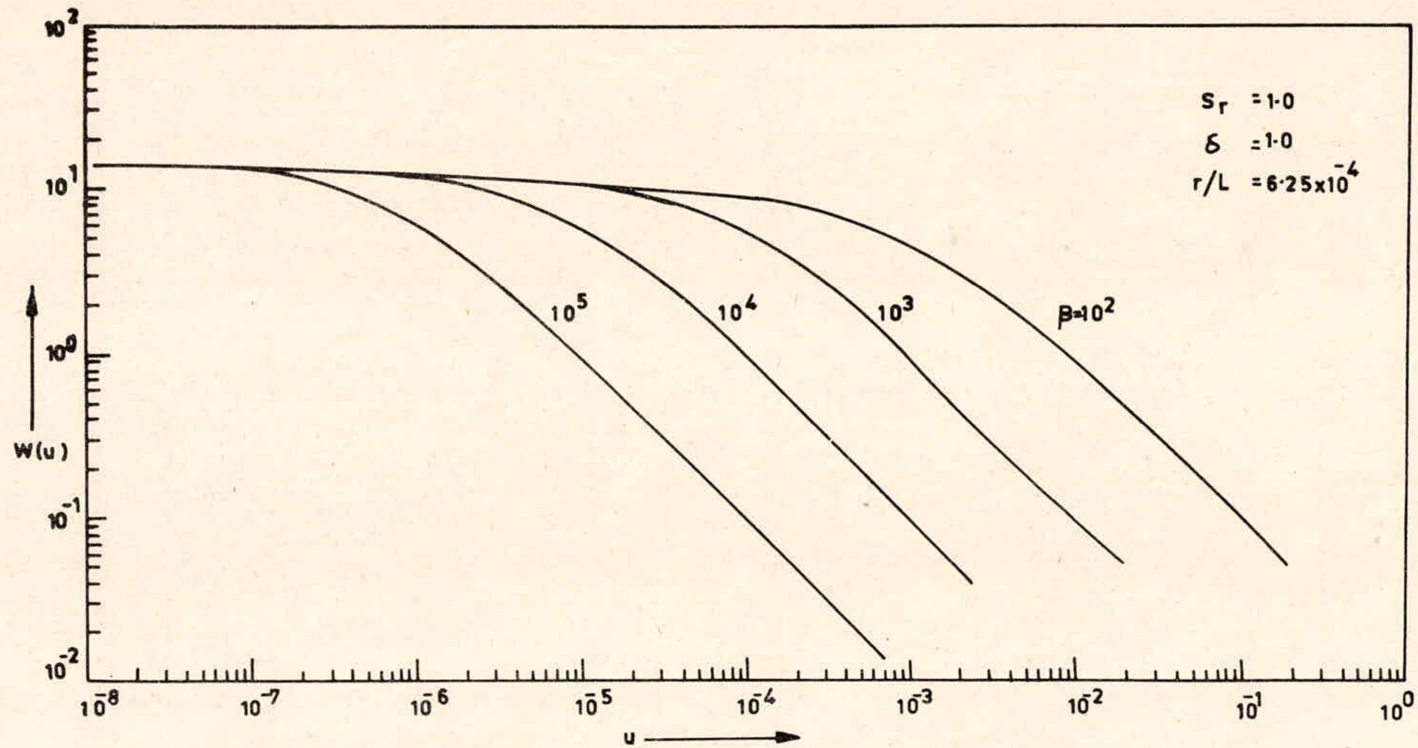

FIG. 8 -TYPE CURVES FOR A FULLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW.

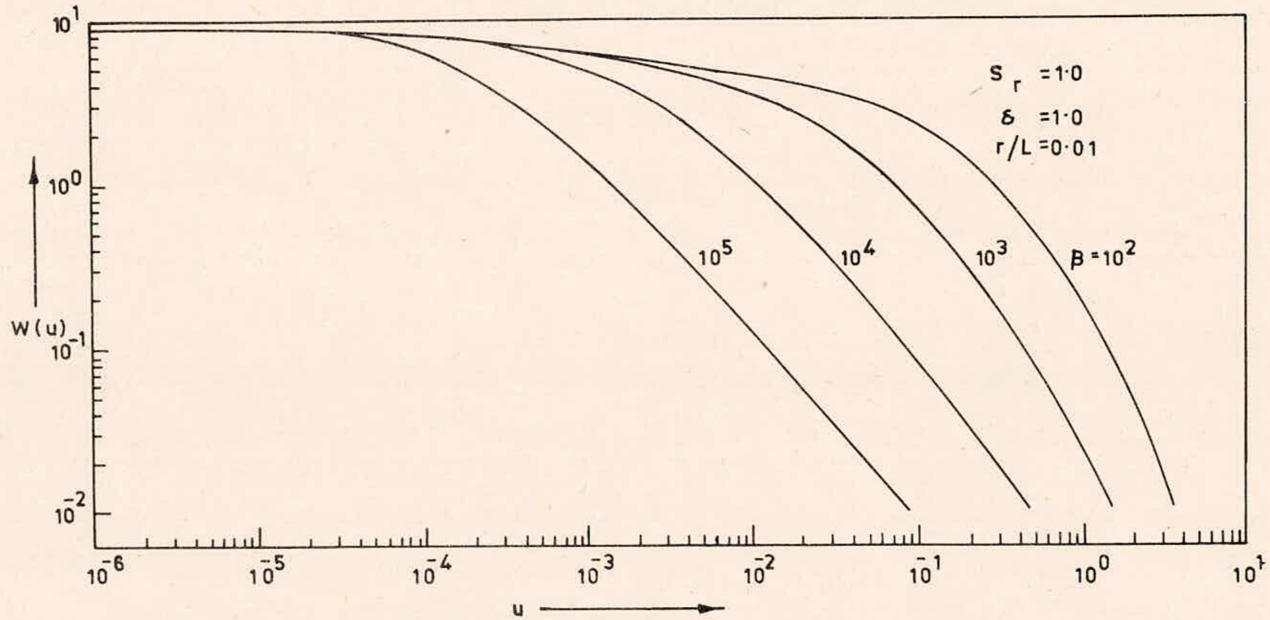

FIG. 9 - TYPE CURVE FOR A FULLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

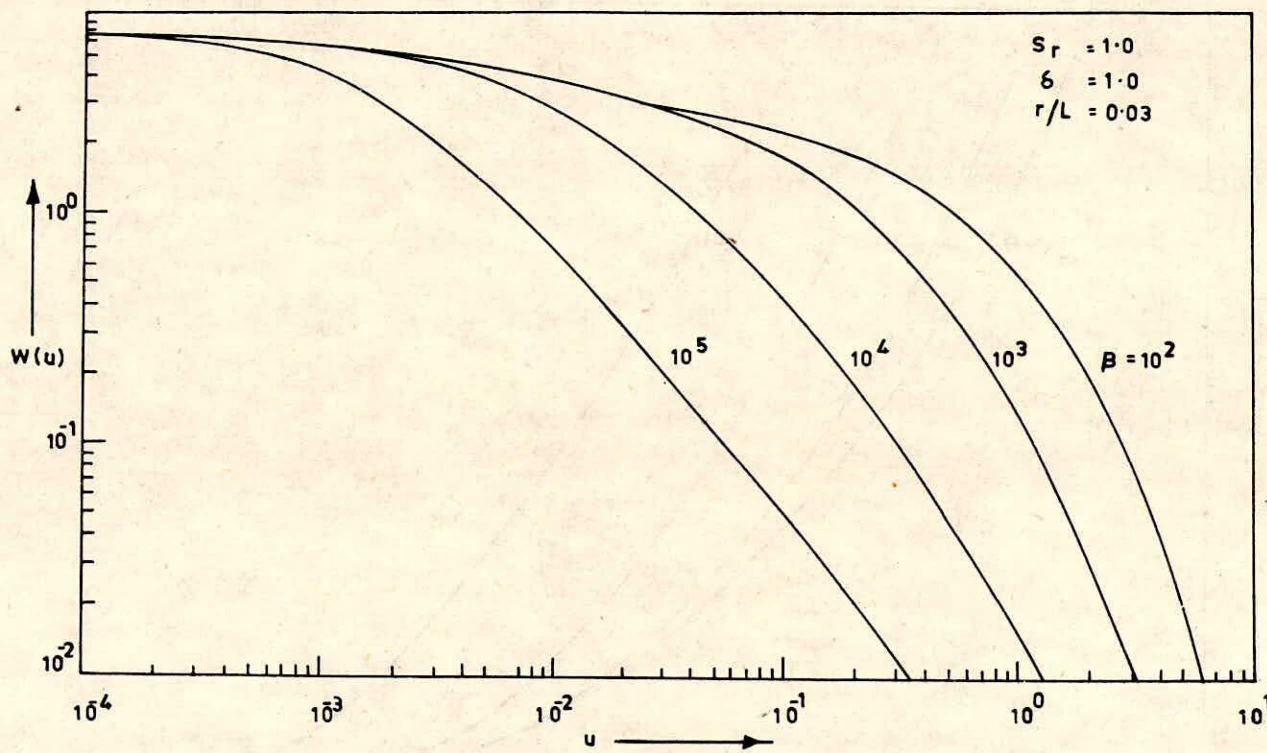

FIG. 10 - TYPE CURVES FOR A FULLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

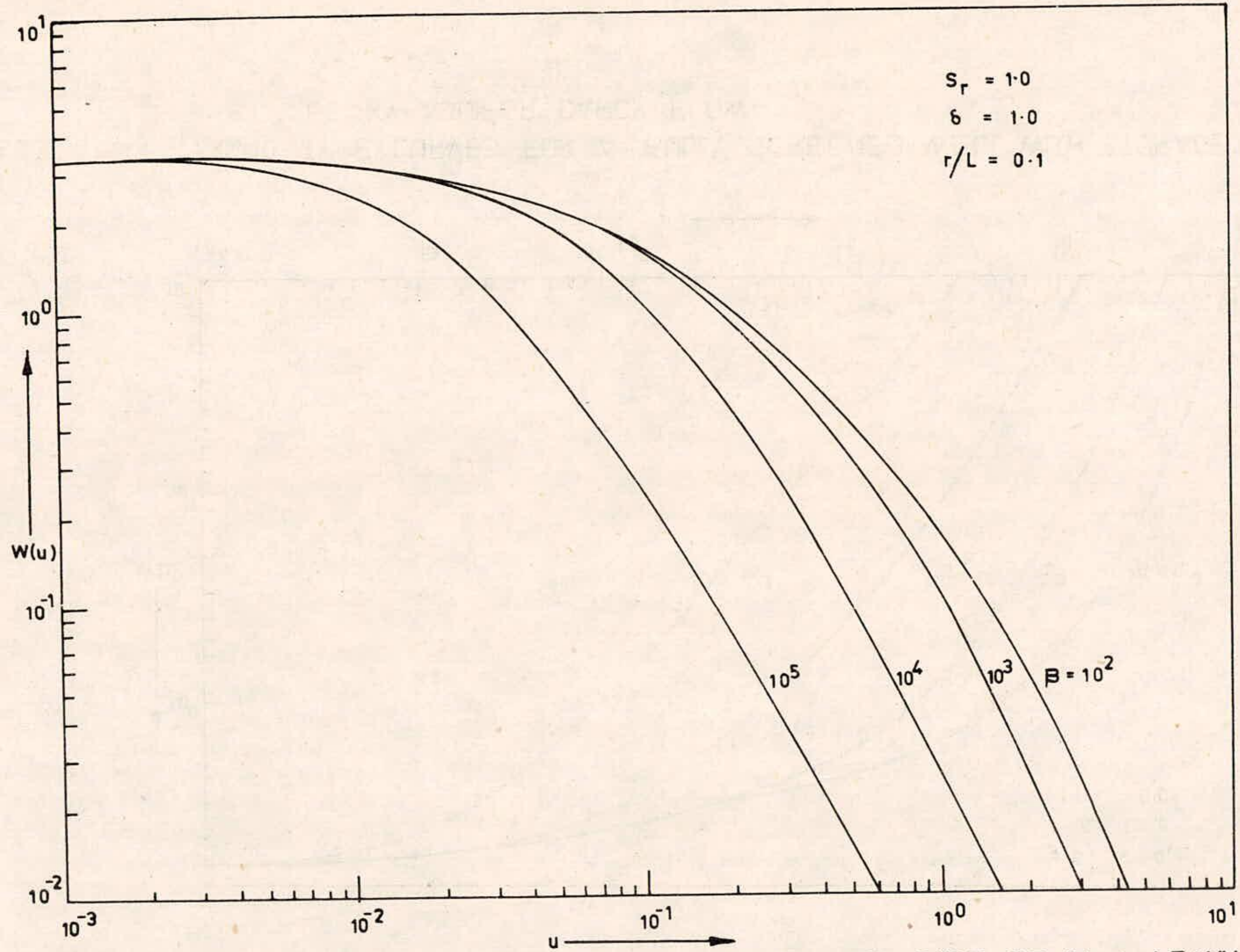

FIG. 11 - TYPE CURVES FOR A FULLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

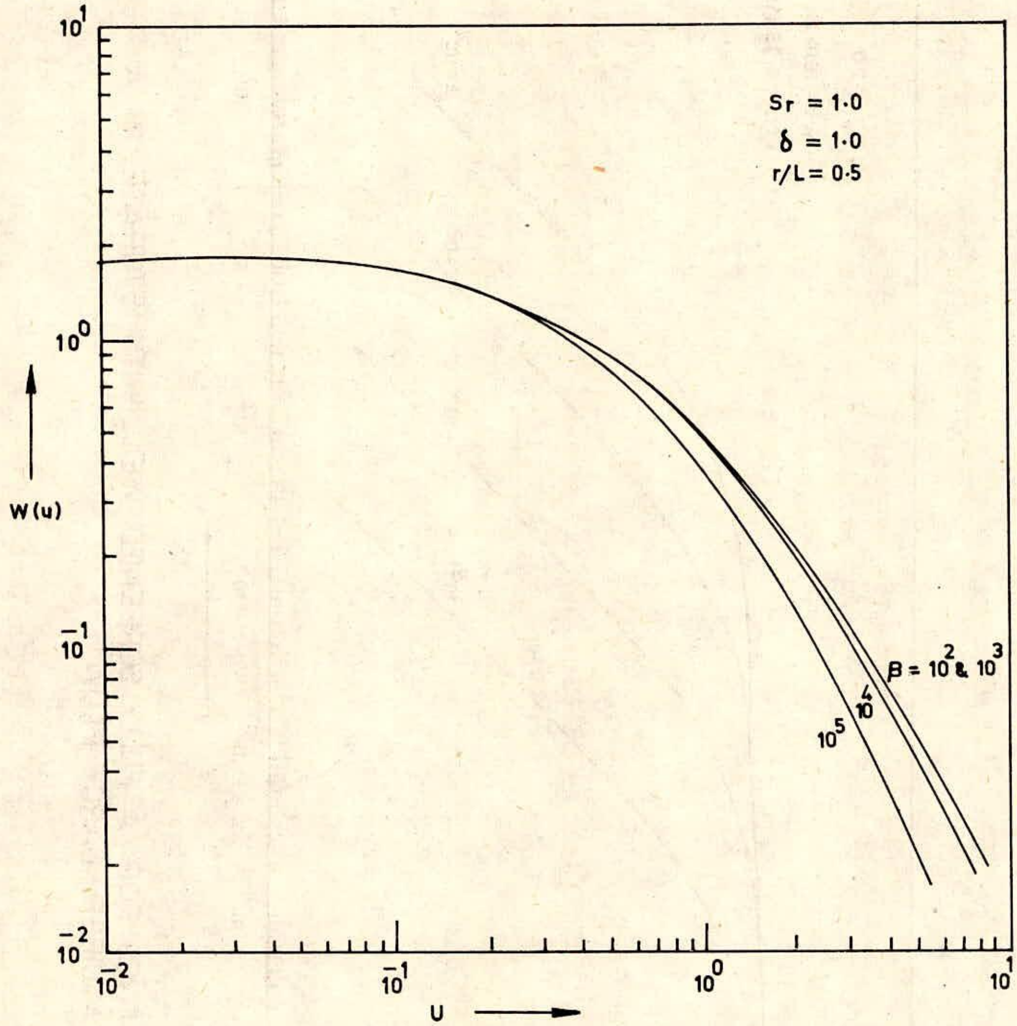

FIG. 12–TYPE CURVES FOR A FULLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER – DARCY FLOW

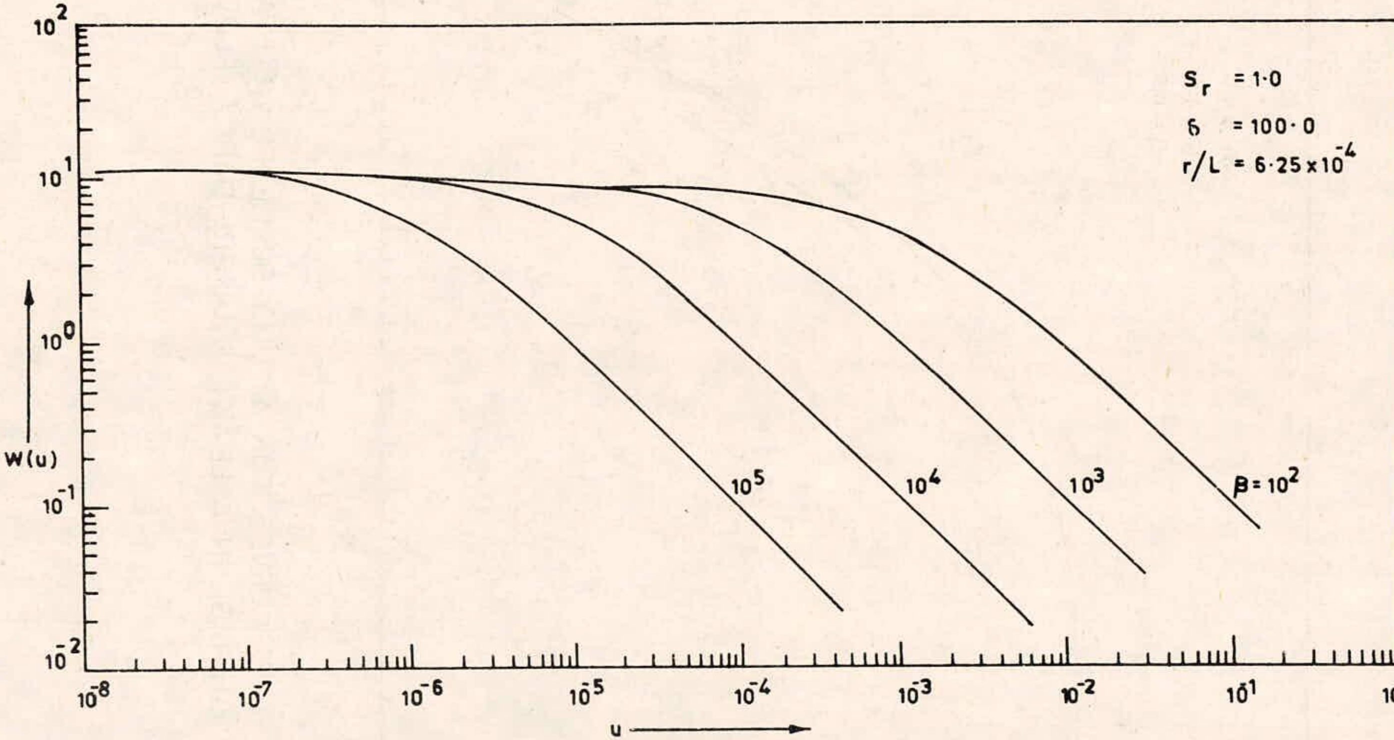

FIG. 13-TYPE CURVES FOR A FULLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

TABLE 1 COMPARISON OF THE RESULTS OBTAINED BY THE FINITE ELEMENT MODEL WITH THE PAPADOPULOS AND COOPER'S VALUES

u_W	$W(u_W, \beta)$			
	β = 1.0 E+02		β = 1.0 E+05	
	Papadopulos and Cooper	FEM	Papadoplulos and Cooper	FEM
1.0 E -07	1.554 E + 01	1.559 E +01	1.513 E +01	1.519 E +01
5.0 E -07	1.393	1.397	1.013	1.017
1.0 E -06	1.324	1.329	6.779 E +00	6.814 E +00
5.0 E -06	1.162	1.166	1.817	1.860
1.0 E -05	1.087	1.091	9.493 E -01	9.518 E-01
5.0 E -05	9.229 E + 00	9.276 E +00	1.975	1.981
1.0 E -04	8.443	8.481	9.932 E -02	9.940 E-02
5.0 E -04	6.031	6.079	1.997	1.999
1.0 E -03	4.545	4.586	9.992 E -03	1.00
5.0 E -03	1.540	1.579	2.000	2.00 E -03
1.0 E -02	8.520 E -01	8.547 E -01	1.000	1.00
5.0 E -02	1.896	1.973	2.000 E -04	--
1.0 E -01	9.666 E -02	9.668 E-02	1.000	--
5.0 E -01	1.974	1.971	2.000 E -05	--

the type curves presented herein include up to r/L ratio of 0.5. The time corresponding to the deviation of the curve from the straightline portion (the junction of Zone I and Zone II) is termed as the deviation time. It has been found that the product of non-dimensional time parameter correspondinc to the deviation time if the observations were taken on well boundary and the storage parameter β remains a constant. Using this value, the transmissivity of the aquifer can be assessed. Also it has been found that the product of non-dimensional time parameter corresponding to merging time (the time corresponding to the junction of Zone II and III) and the storage parameter β has a specific value which depends only on r/L ratio. Knowing the value of this product the domain where the effects of well storage are predominant can be assessed. However, for S' = S, the type curve for r/L = 1.0 is also presented (Figure 14). This merging of the type curves for 'wells with storage' with those of 'no storage' curves at a short distance in a leaky aquifer compared to a confined case can be attributed to the fact that the effects of storage and the effects of leakage are cumulative. The aquitard storage brings about a reduction in the drawdown in the aquifer as in the case of ' well with no storage'. This becomes evident from a plot between u versus W(u) for $\beta = 10^3$ at r/L = 0.01 for S' = S and S' = 100S (Figure 15). This reduction in drawdown has further reduced the merging distance in the case of δ = 100. It is observed that for $\beta = 10^5$ (the extreme storage parameter used in this investigation) at r/L = 0.5, the type curve differed from the type

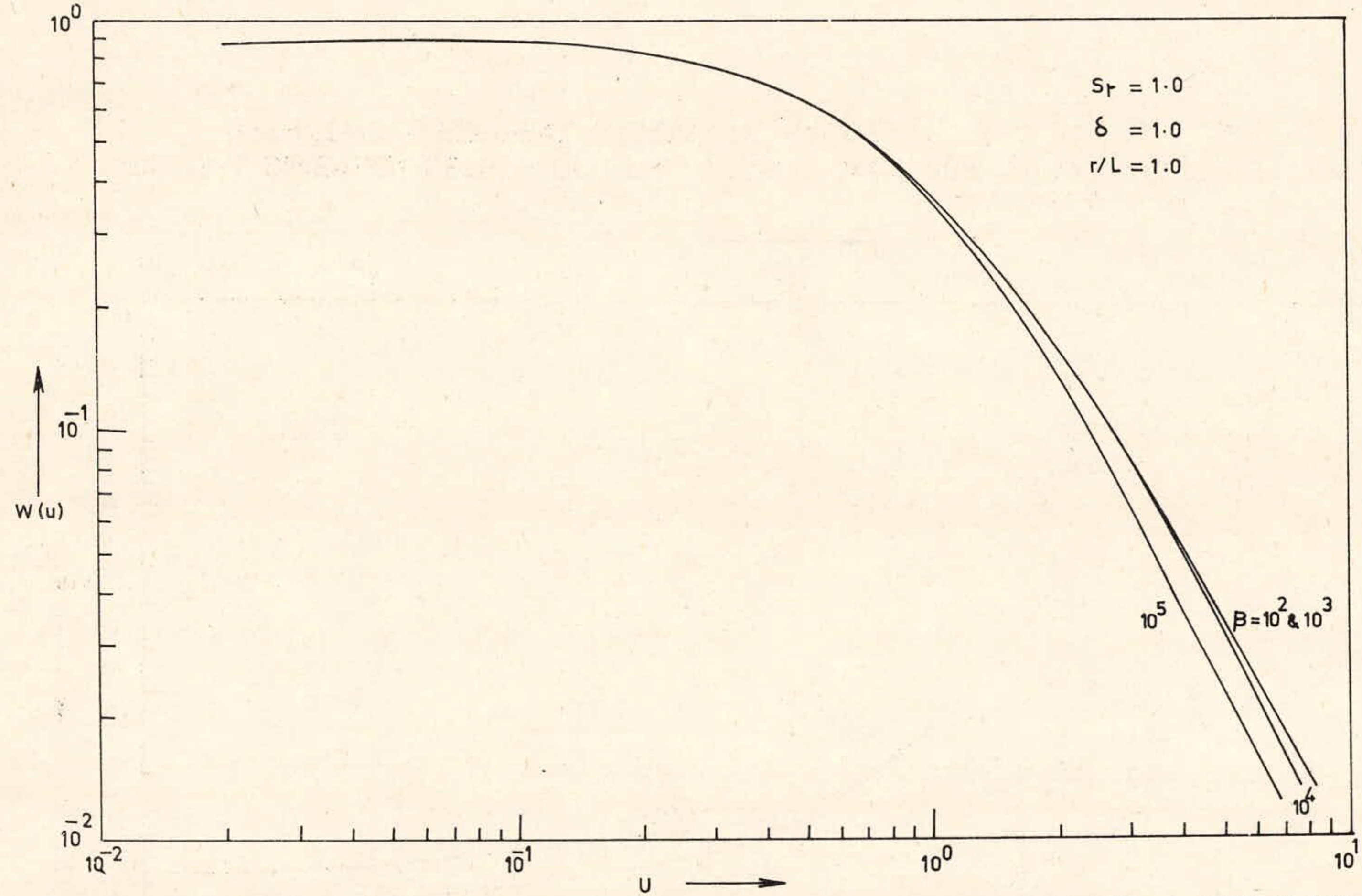

FIG. 14 - TYPE CURVE FOR A FULLY SCREENED WELL IN LEAKY AQUIFER - DARCY FLOW

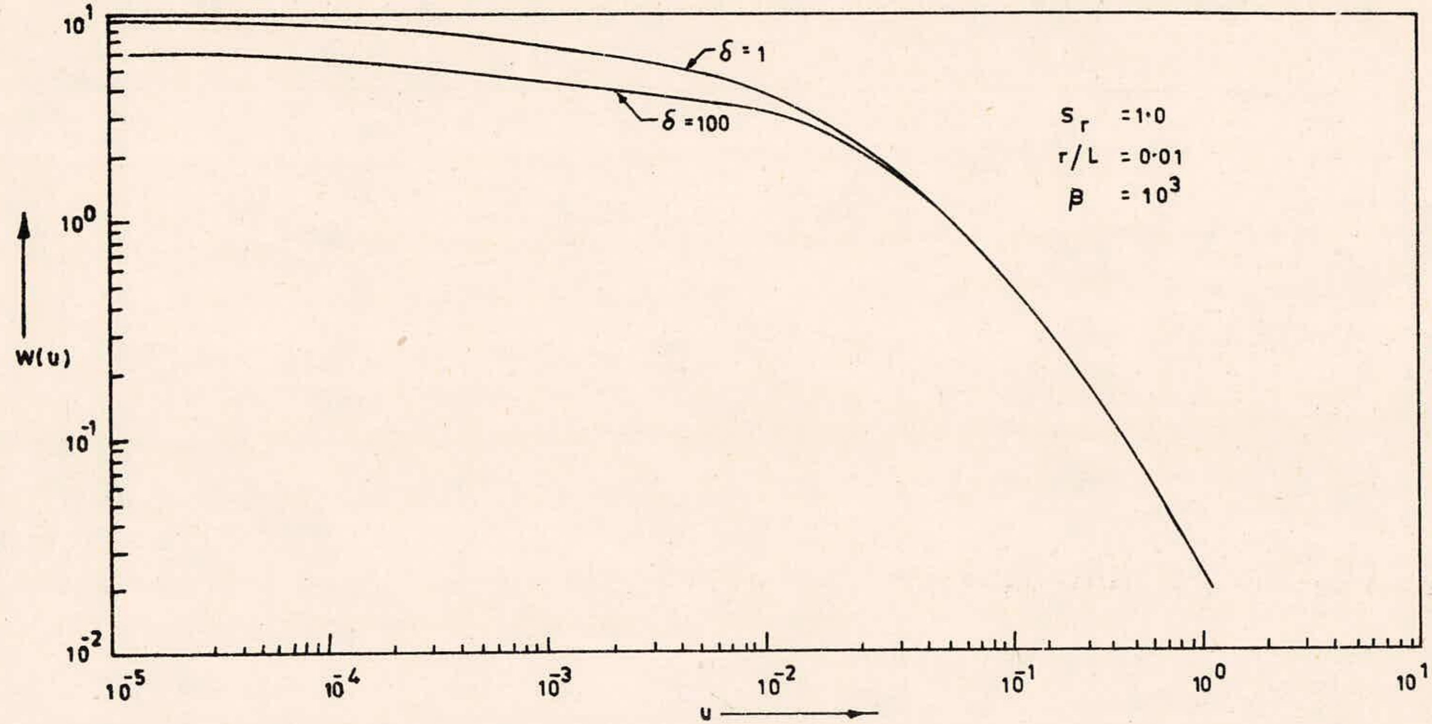

FIG. 15-A DIMENSION LESS PLOT OF u VERSUS W(u) FOR LEAKY AQUIFERS (WITH TWO DIFFERENT STORATIVITY RATIOS)

curve for ' well with no storage' by approximately 3.5%. Hence, the type curves for r/L = 0.5 and for = 100 are not presented.

5.4.2 Deviation time

In leaky aquifers also, the product $u_{wd} \cdot \beta$ (where u_{wd} is the nondimensional time corresponding the deviation of the type curve from the straight line portion) is found to be the same as in the case of non leaky aquifers and is equal to 2.0. Hence, the expression for t_d (time of deviation) is $t_d = 0.125\ r_c^2/T$ or $t_d = C_d \cdot r_c^2/T$ where $C_d = 0.125$. Using this expression, the transmissibility of the aquifer can be determined from the early time pumping history.

5.4.3 Merging time

a) on the well boundary

It is observed that the product of u_m, (the nondimensional time parameter corresponding to the merging time) and the storage parameter, β remains constant in leaky aquifers also. But, it is observed that this product varies with the ratio of the storativities of aquitard and aquifer and r/L ratio. As S'/S increases the product $u_m \cdot \beta$ also increases. In other words, for a particular β, the value of u_m increases with increase in storativity ratio (S'/S) (i.e., the merging time decreases). The values of $u_m \cdot \beta$ for S'/S = 1 and for 100, and the corresponding coefficients, c_m are presented in Table 3

b) For the interior of the aquifer

The product of $u_m \cdot \beta$ is determined for the interior of aquifer (i.e., for different r/L ratios) and for both the storativity ratios and the values are presented in the Table 3 Using these values, the coefficients c_ms can be computed. From these values, it can be said that as r/L ratio increases (for both cases of S'/S) the product $u_m \cdot \beta$ increases (but remains same for different β_s, as in the case of non leaky aquifers) and hence, results in the decrease of the merging time.

5.4.4 Merging distance

As is seen from the type curves presented for r/L = 1.0 it can be concluded that beyond the radial distance equal to the leakage factor the storage effects can be neglected. Quantitatively, it is assessed that by ignoring the well storage effects at r/L = 1.0, the error caused is not more than 5% for $\beta = 10^5$ and less for $\beta \leq 10^5$. Hence, in the present analysis type curves are presented up to r/L ratio of 0.5 for $\delta = 1$ and for $\delta = 100$, they are given up to r/L of 0.1 (for $\delta = 100$, the merging distance is further reduced).

In conclusion, it can be said that the storage effects and effects due to leakage are cumulative and reduces the drawdown for a particular discharge. This reduction in the drawdowns increases with increase in the storativity ratio. Consequently, the merging time and the merging distance also reduce. However, as in the case of nonleaky aquifers, the deviation time remains unaltered.

5.5 Partially Screened Wells with Storage in Leaky Aquifers

For this analysis, twentyfour cases are solved and analysed. The geometric and hydrogeological properties adopted for these cases are shown in Table 2. Three different values of screening ratios, S_r , are choosen, viz., 0.75,0.50 and 0.25 for each storativity ratio,S'/S, (i.e., S'/S = 1 and 100). The storage parameter, β is varied from 10^2 to 10^5 for each of the above mentioned cases. Plots are drawn on a log-log scale for different β values between the nondimensional time parametner, u, and the non-dimensional drawdown,W(u). Such plots are presented herein for five different r/L ratios and for different S_r s ,(Figure 16 to 33). In this analysis, finiteness of the well is taken into account.

5.5.1 Shape of the type curves

The general pattern and shape of the type curves for partially screened wells (Figures 16 to 33) did not differ from those for fully screened wells in leaky aquifers, except that the drawdowns obtained in these cases are more. These drawdowns increase with decrrease in the screening ratio but decrease with increase in storativity ratios. The three regions, which are discussed in sub-section 5.4.1 are seen distinctly.

5.5.2 Deviation time

The deviation time, as it signifies, is the time at which the storage from the aquifer and aquitard system commences to subscribe to the well discharge. The type curve,which is a straight line until then, deviates from it and has a variable curvature till it meets the 'no storage' curve. The product of $u_{wd} \cdot \beta$ is found to remain constant and is independent of screening ratio, the storativity ratio, with its value equal

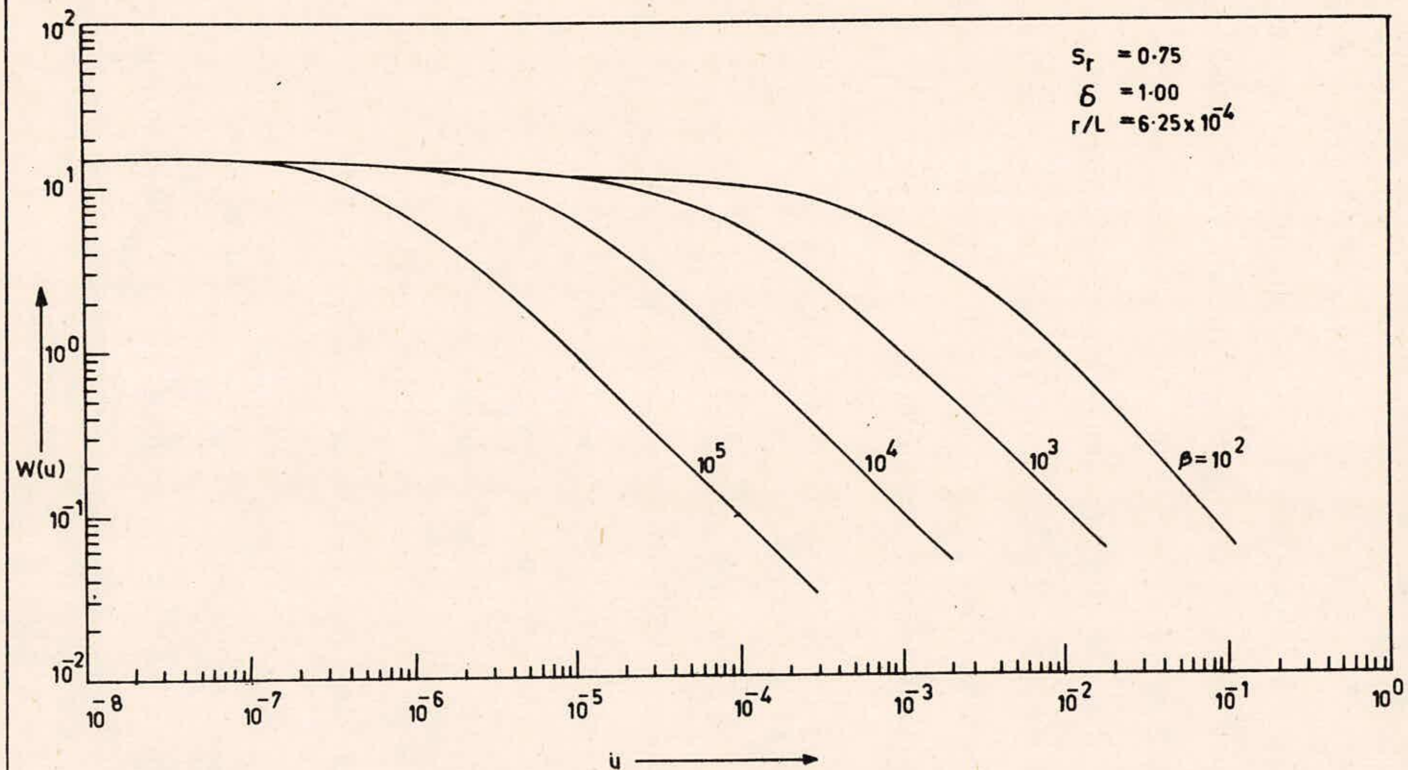

FIG 16 – TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER – DARCY FLOW

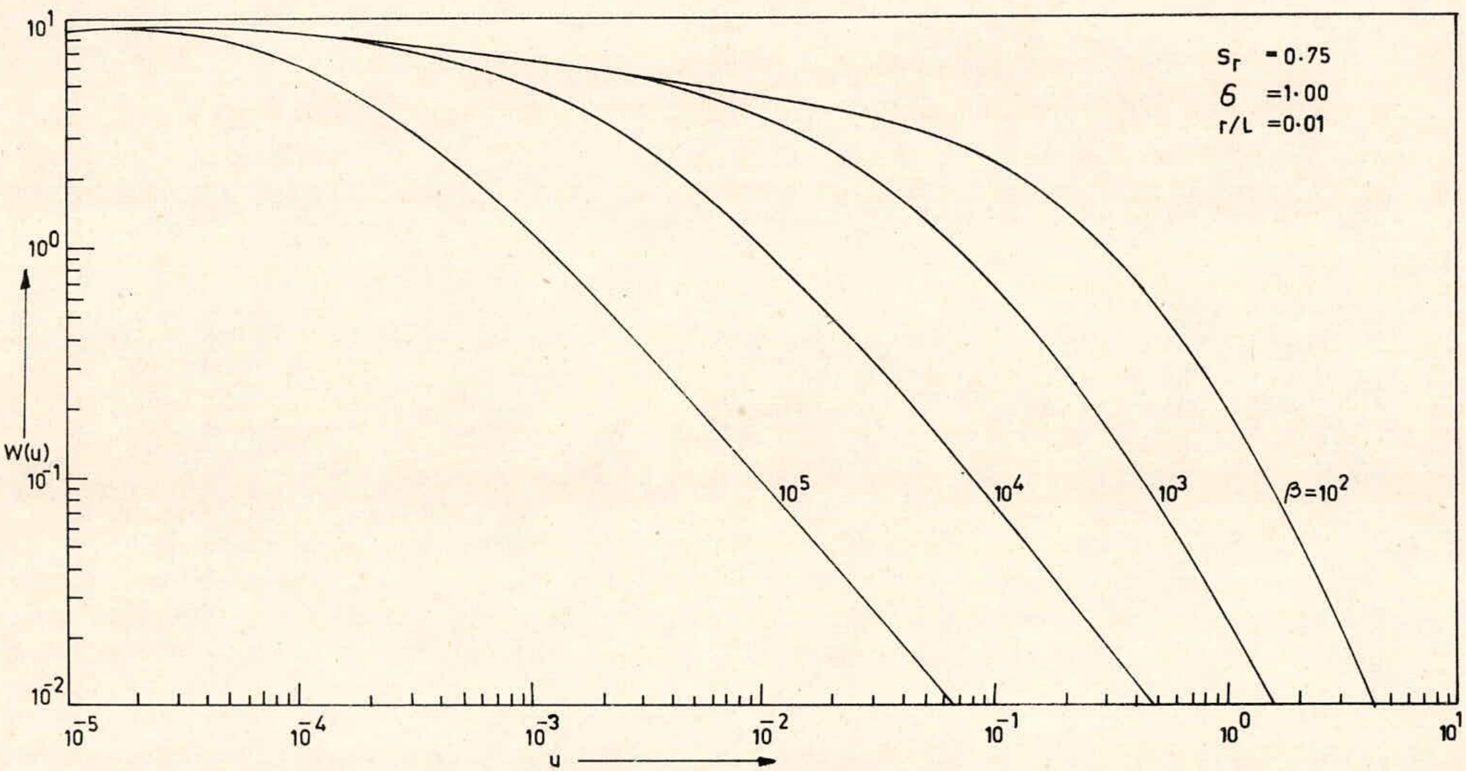

FIG 17 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER-DARCY FLOW

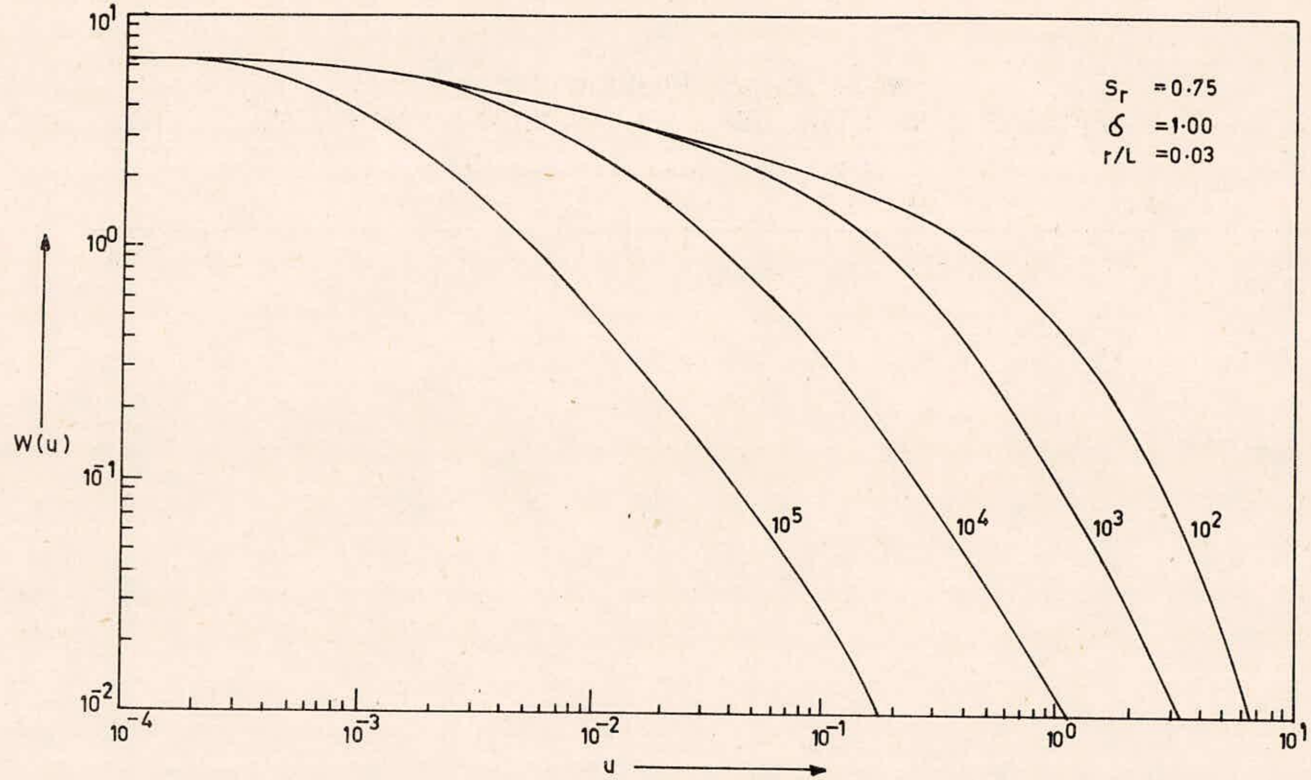

FIG 18 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER-DARCY FLOW

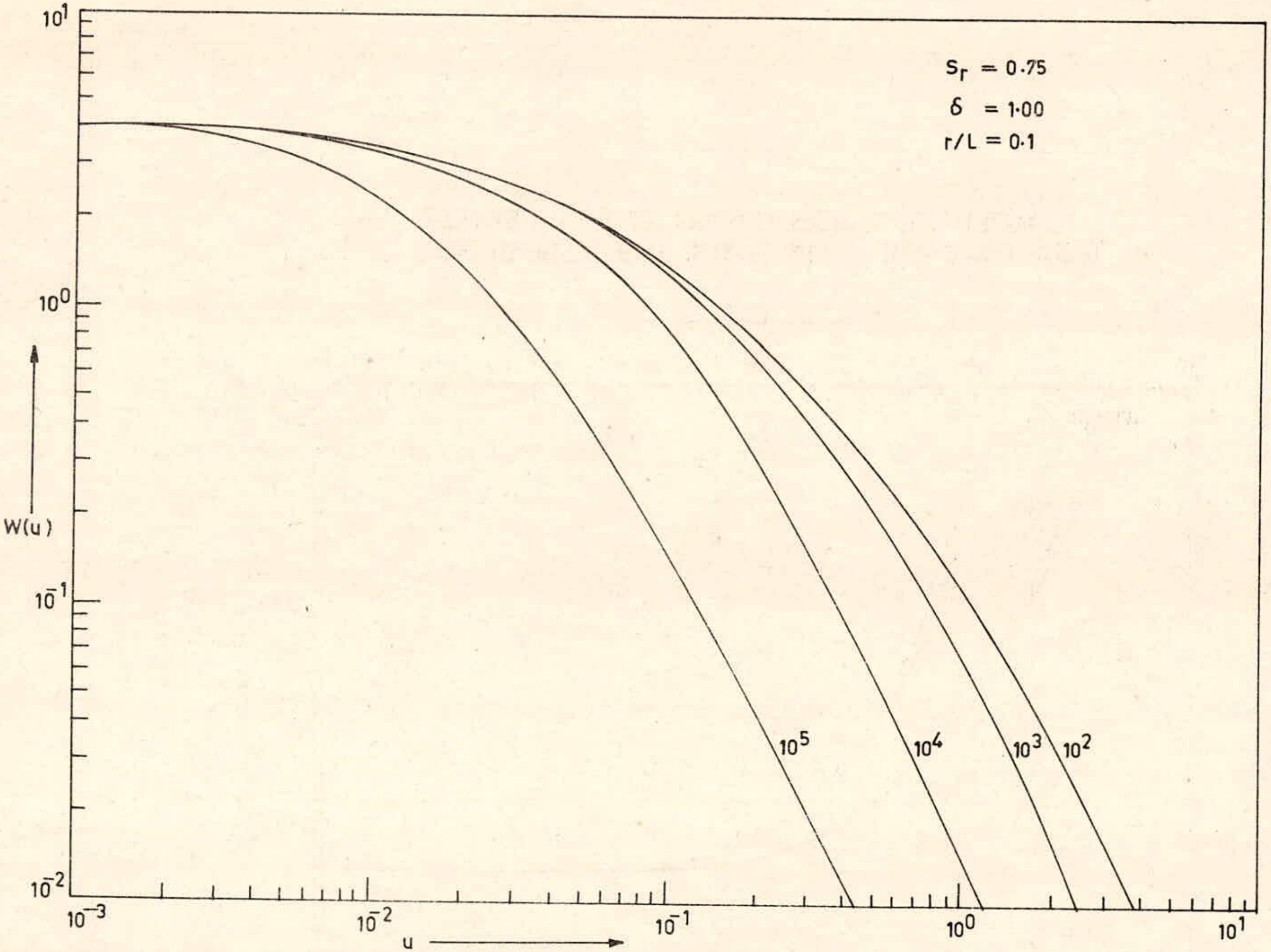

FIG. 19 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER DARCY FLOW

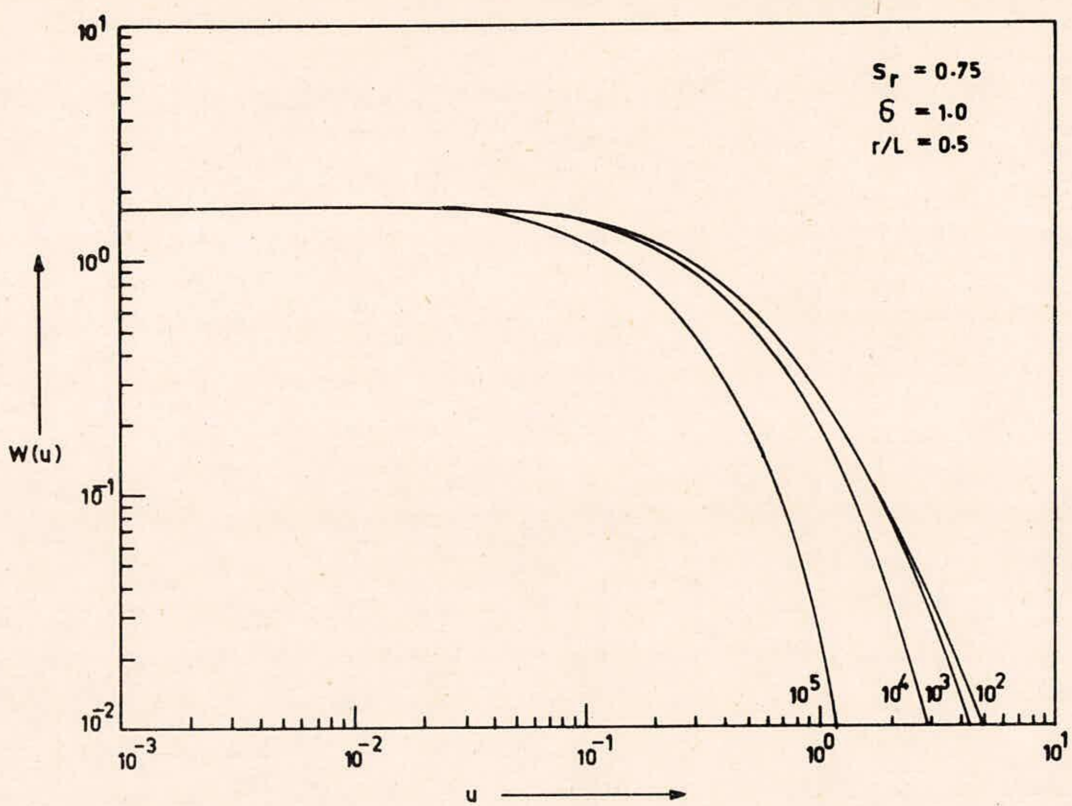

FIG. 20 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER-DARCY FLOW

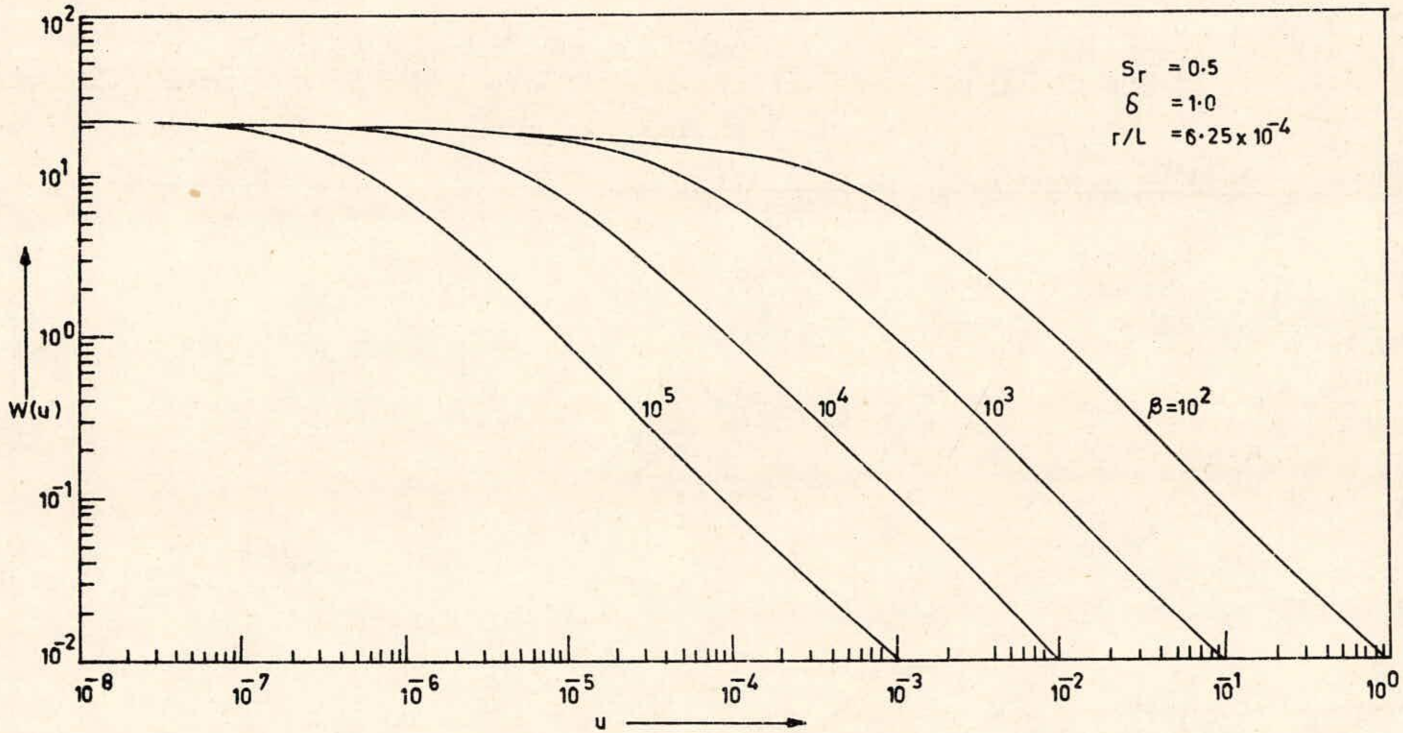

FIG. 21 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER-DARCY FLOW

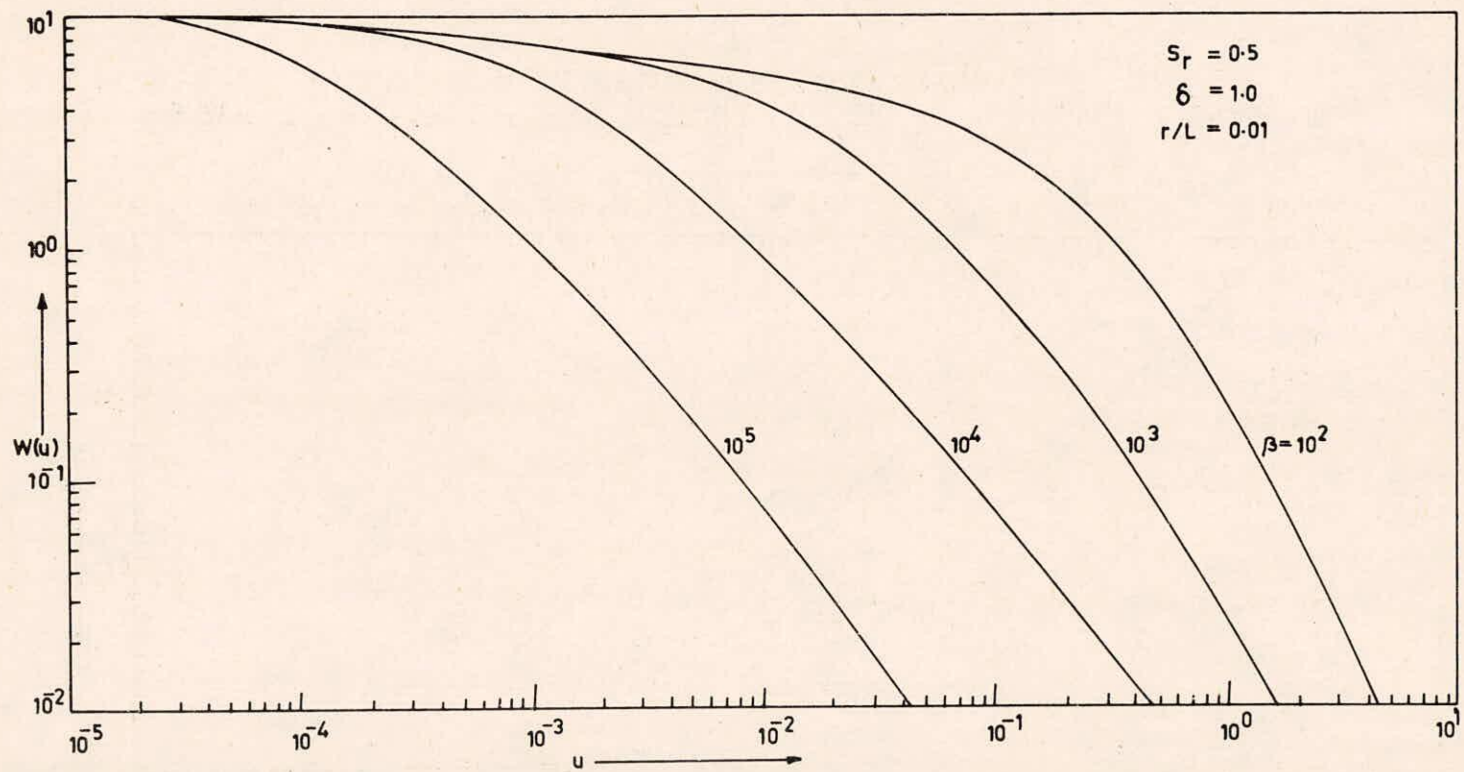

FIG. 22 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER DARCY-FLOW

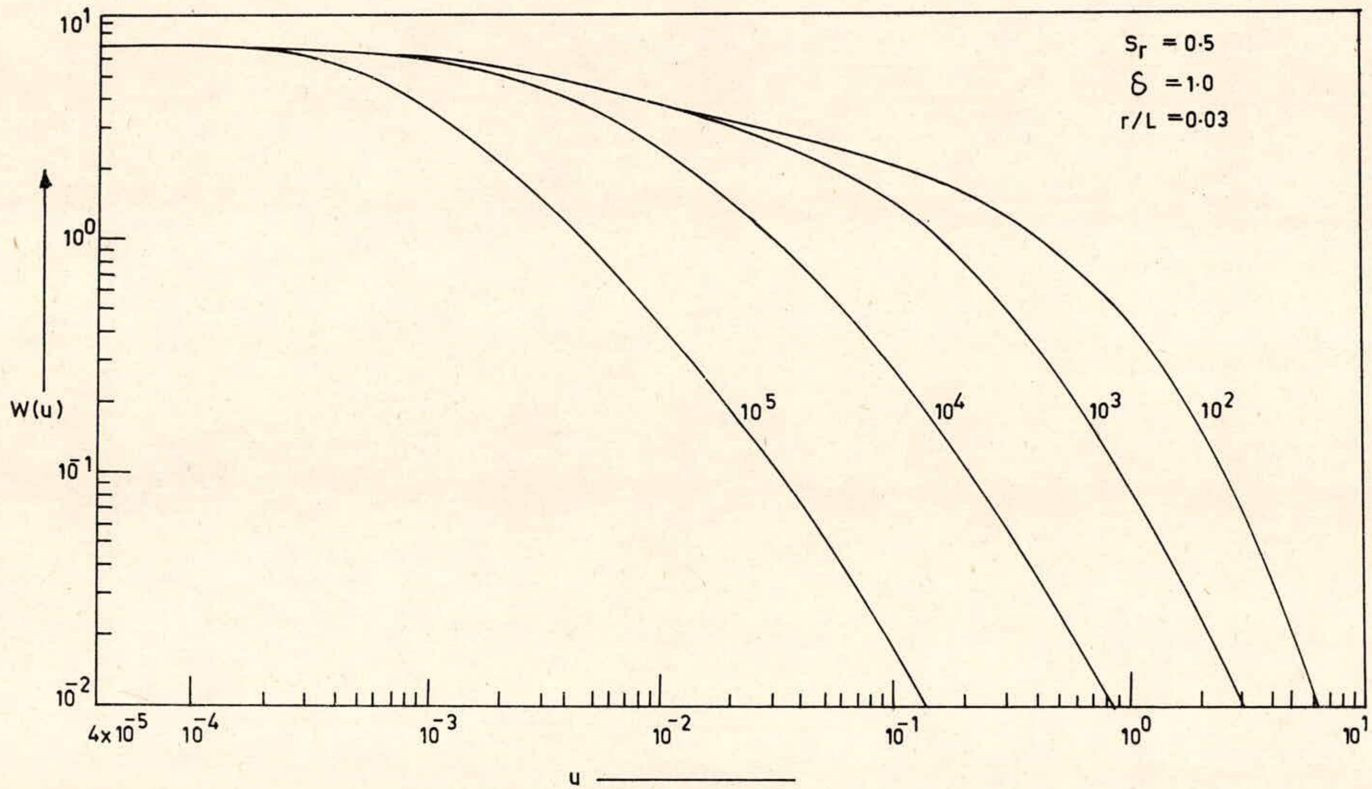

FIG. 23 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER–DARCY FLOW

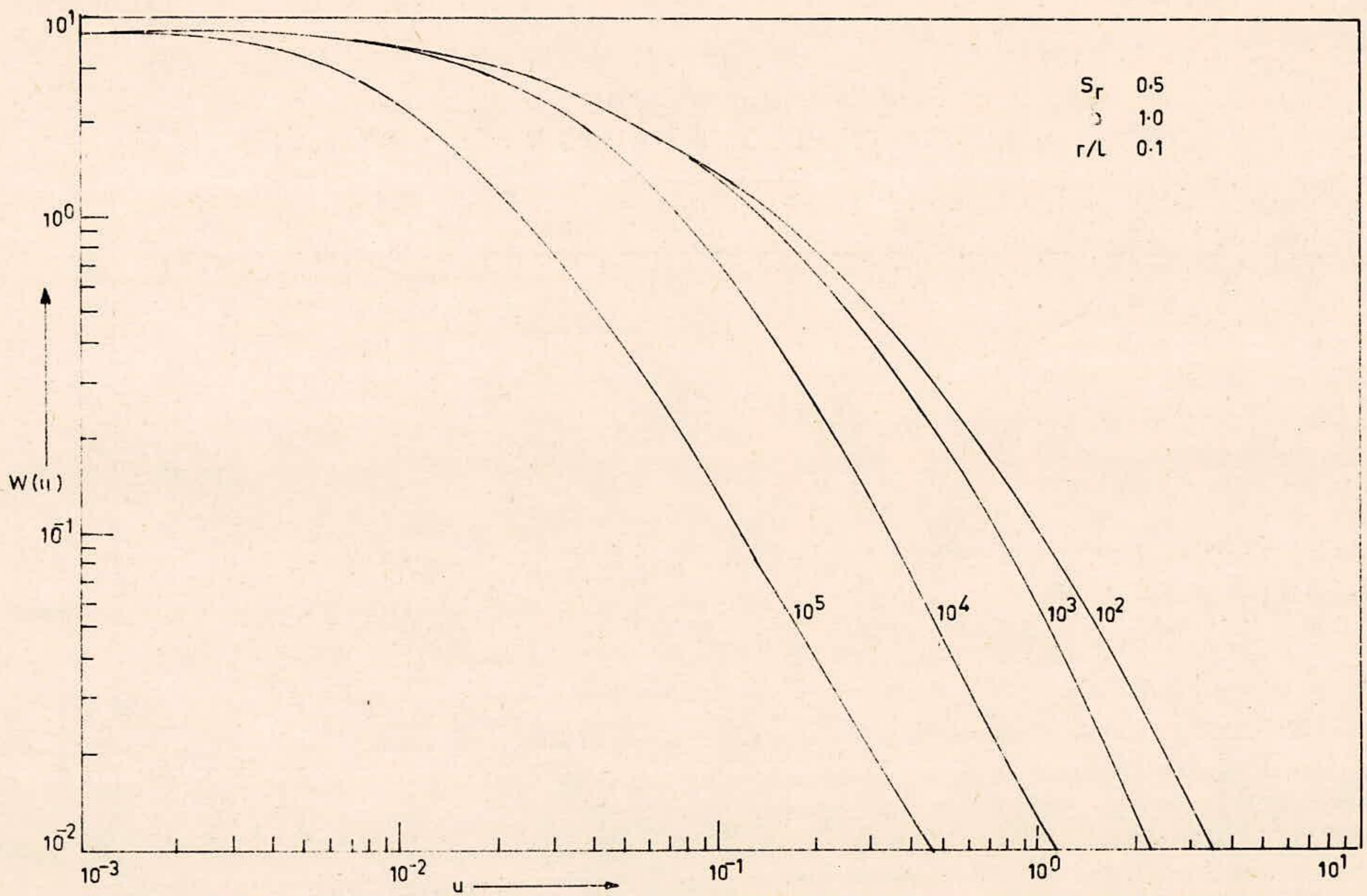

FIG 24 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER-DARCY FLOW

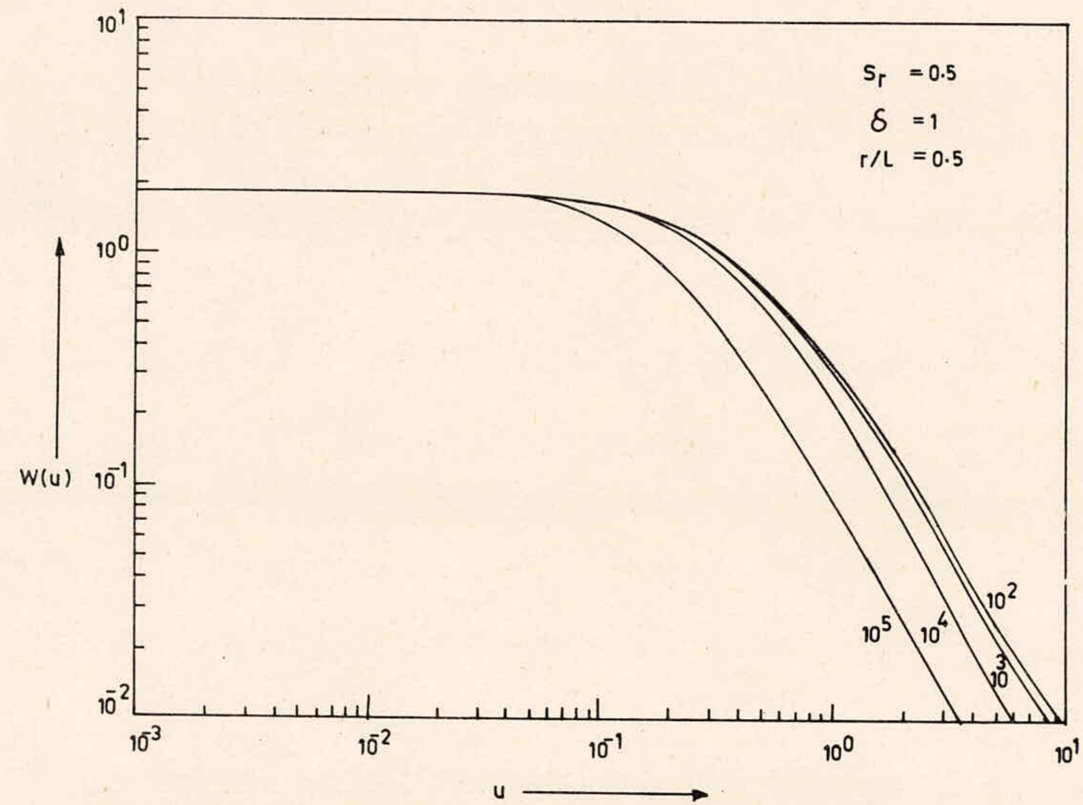

FIG. 25 TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER-DARCY FLOW

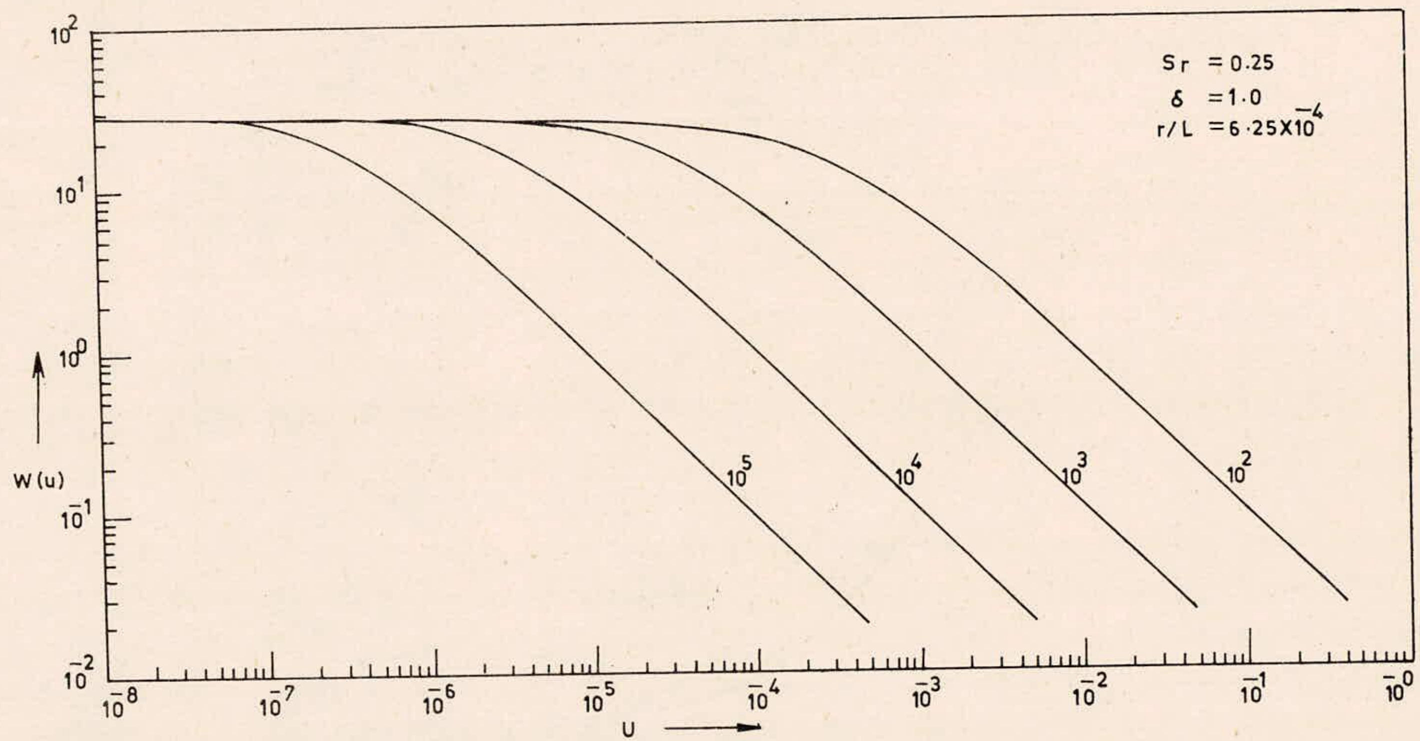

FIG. 26 - TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

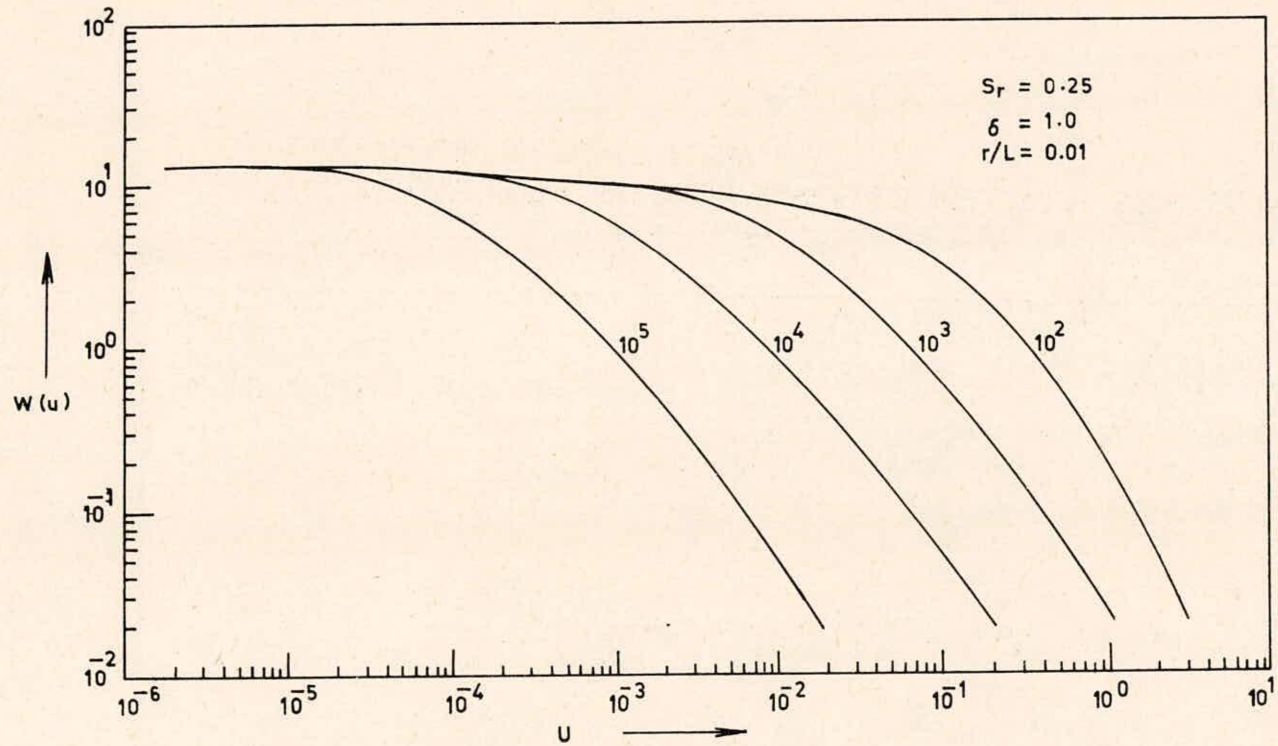

FIG. 27-TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

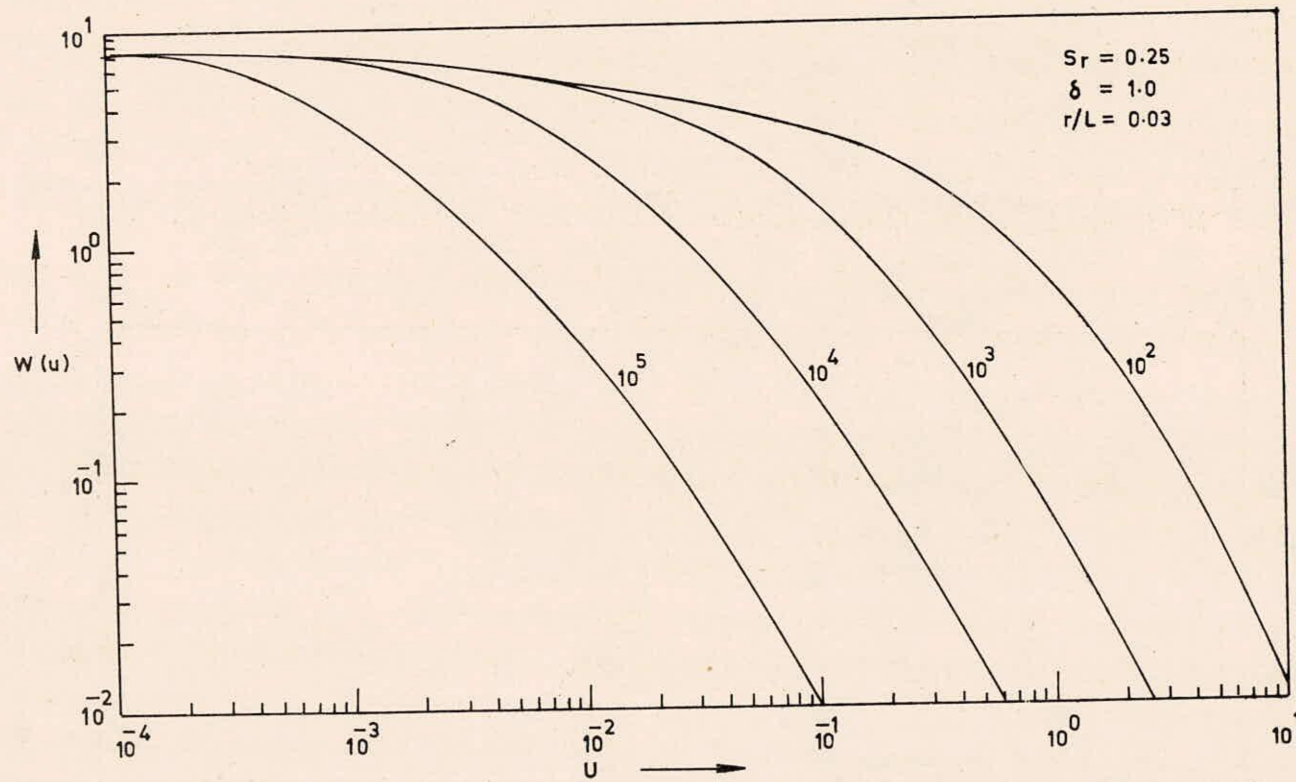

FIG. 28-TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

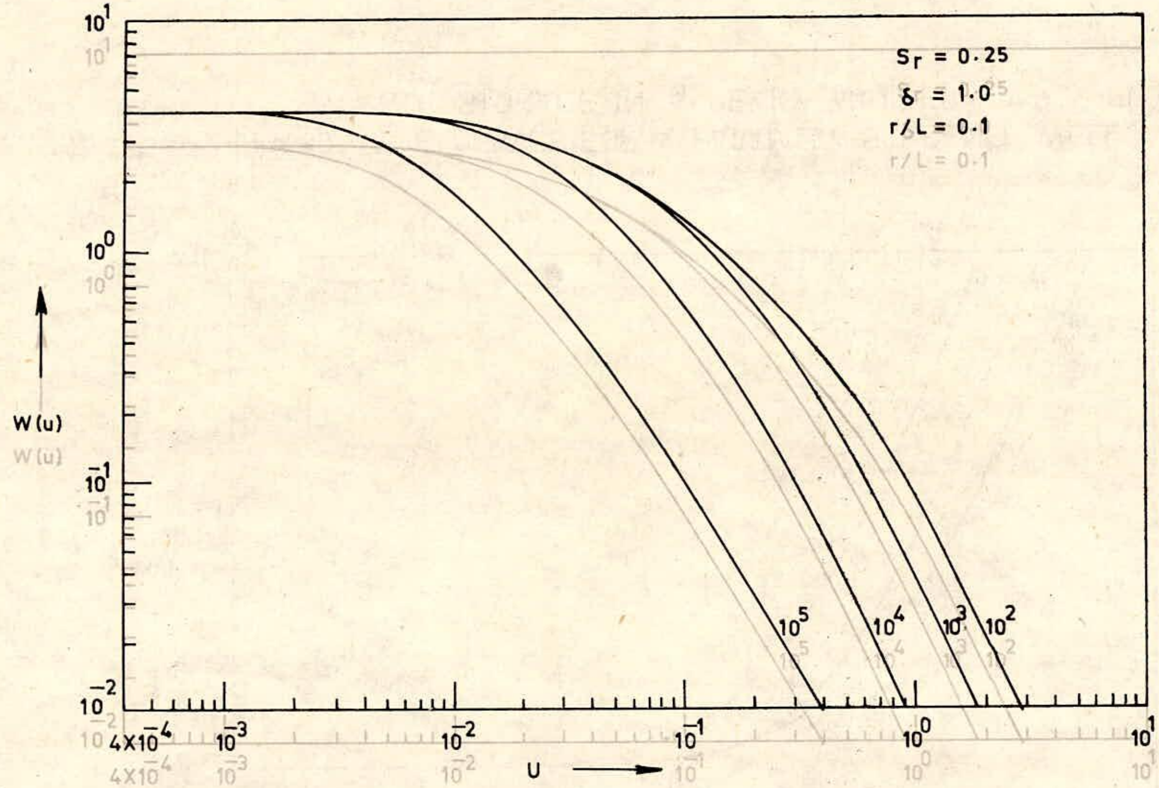

FIG. 29-TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

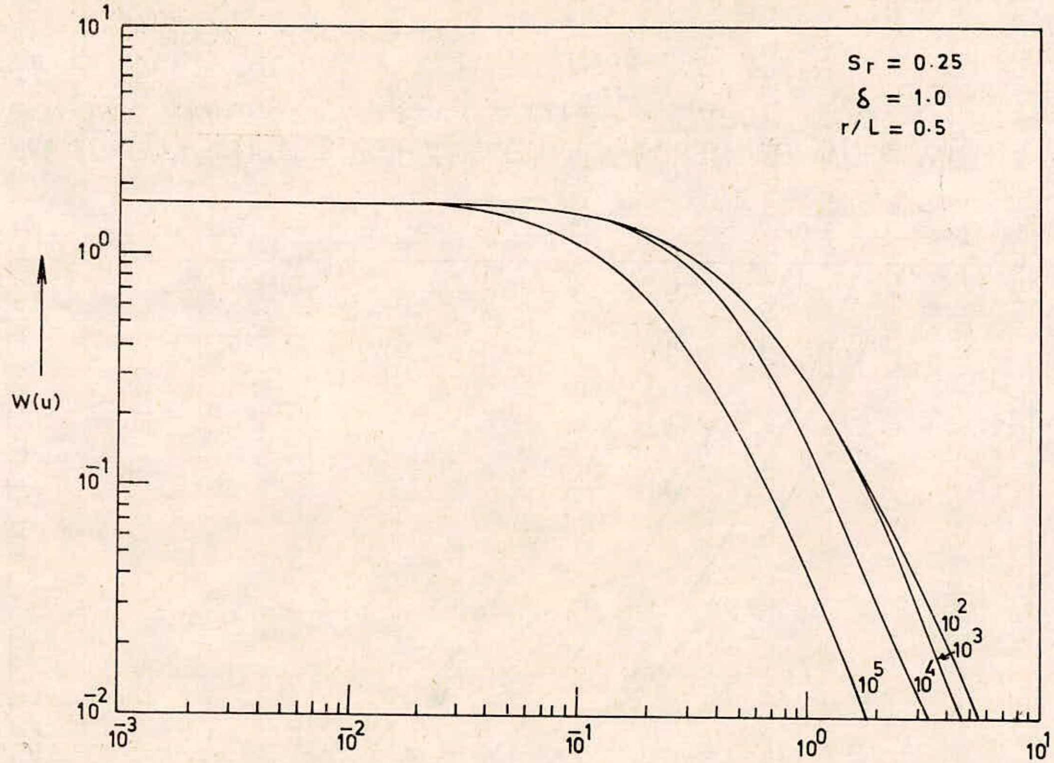

FIG. 30-TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER-DARCY FLOW

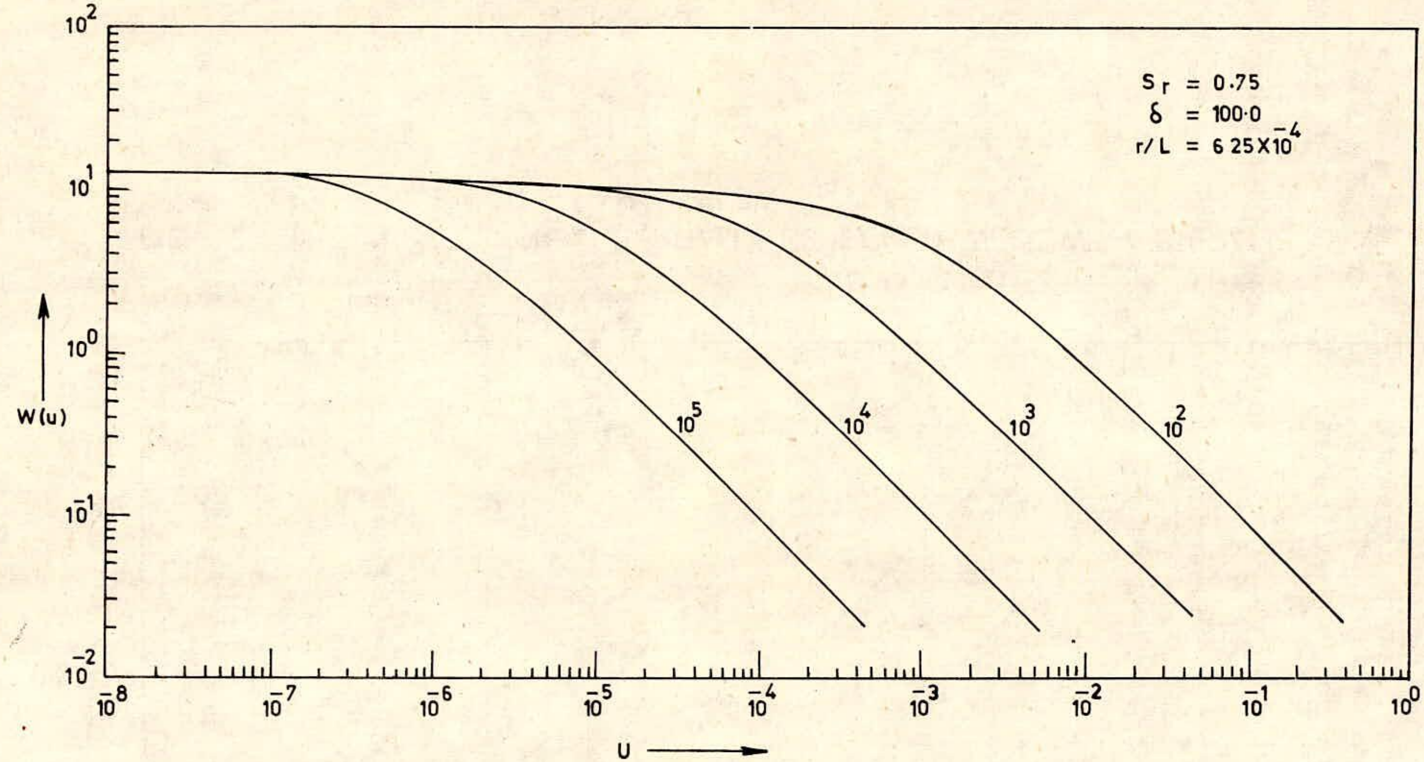

FIG. 31 – TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER – DARCY FLOW

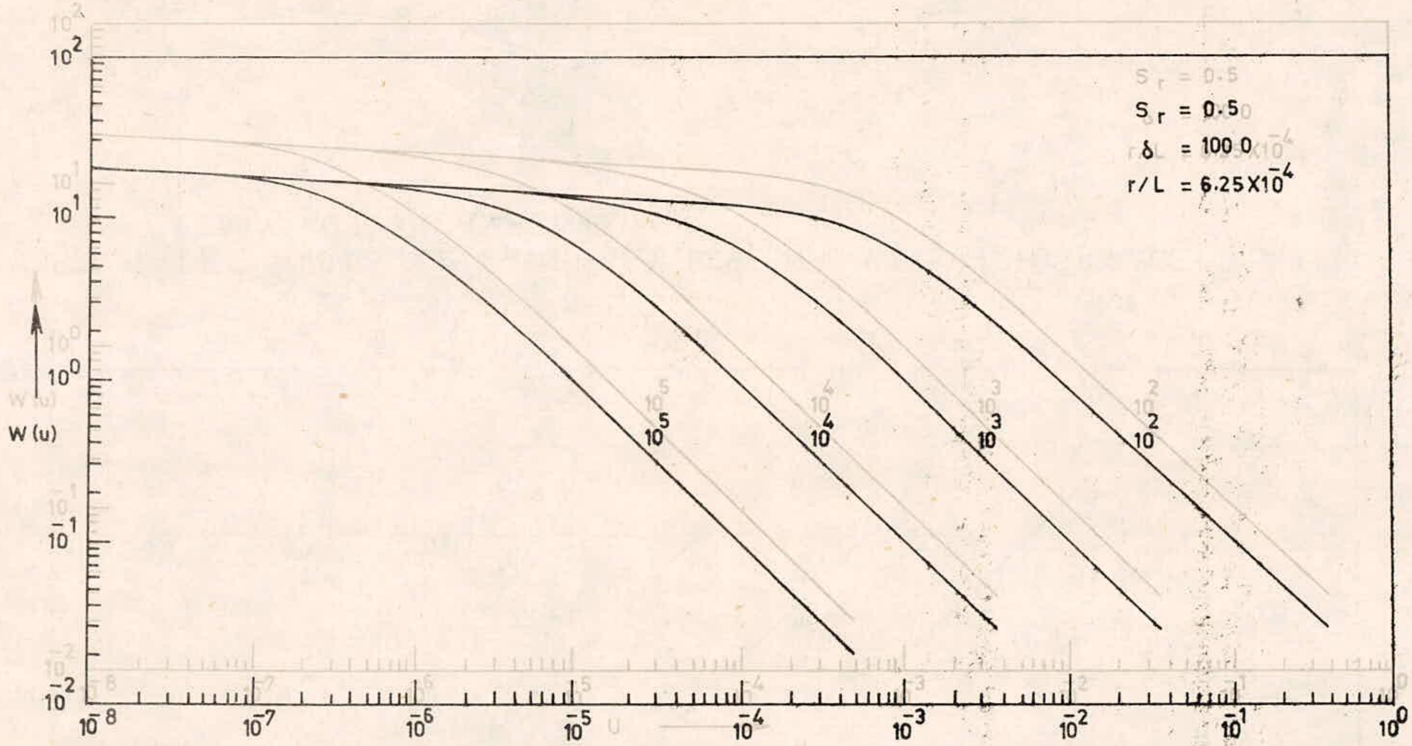

FIG. 32-TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

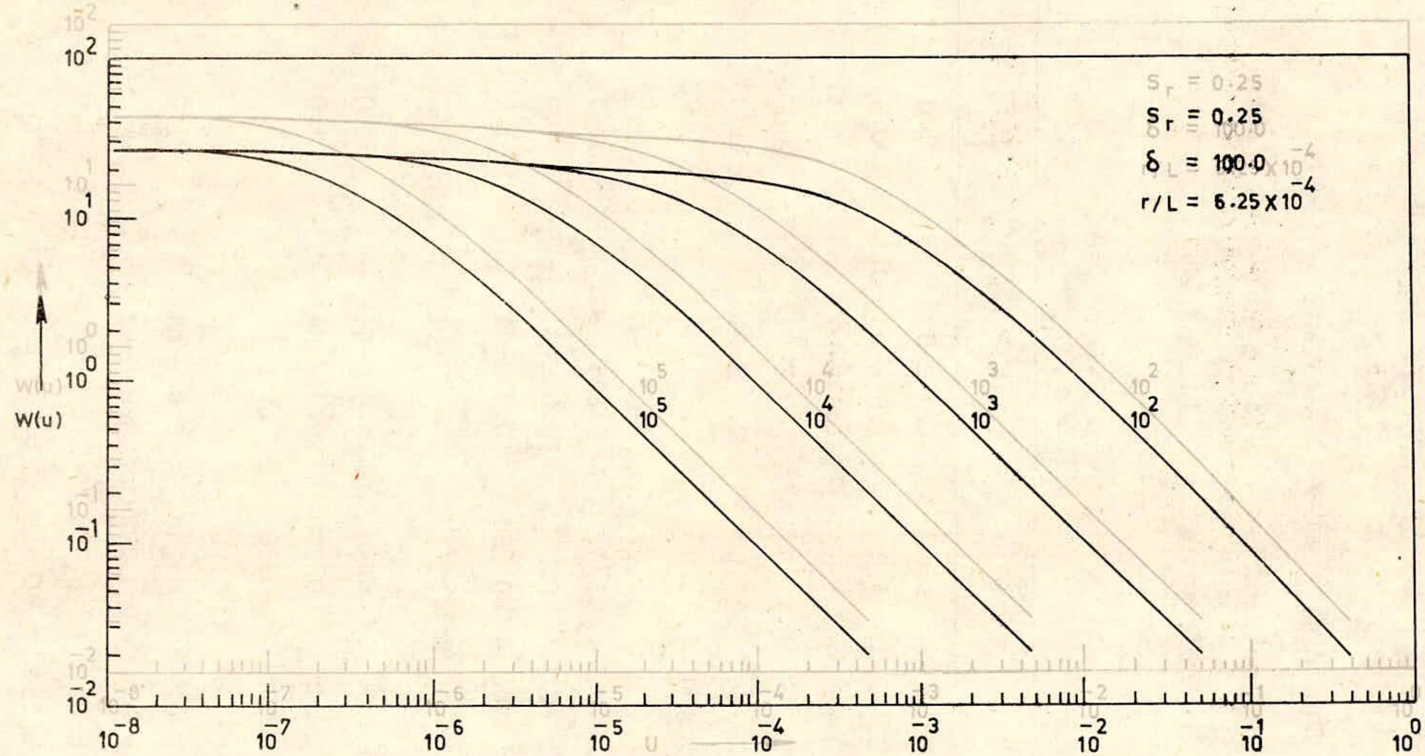

FIG. 33 - TYPE CURVES FOR A PARTIALLY SCREENED WELL WITH STORAGE IN A LEAKY AQUIFER - DARCY FLOW

TABLE - 2 SOLVED CASES OF FULLY PARTIALLY SCREENED WELLS IN LEAKY AQUIFERS

Radius of the Pumping Well, r_w = 0.076 m

Thickness of the Aquifer, m = 12.25 m

Thickness of the Aquitard, m' = 12.25 m

Permeability of the Aquifer, K = 440 m/day

Permeability of the Aquitard, K' = 440 m/day

Constant discharge drawn from the well, Q = 0.046 m^3/sec

Case No.	r_c	β	S_r	S_s	$\delta = S'_s/S_s$
FDS-1	0.5000	1.0 E +02	1.0	1.0 E-03	1.00
FDS-2	0.5000	1.0 E +02	1.0	1.0 E-03	100.00
FDS-3	1.5812	1.0 E +03	1.0	1.0 E-03	1.00
FDS-4	1.5812	1.0 E +03	1.0	1.0 E-03	100.00
FDS-5	2.5000	1.0 E +04	1.0	2.5 E-04	1.00
FDS-6	2.5000	1.0 E +04	1.0	2.5 E-04	100.00
FDS-7	3.7500	1.0 E +05	1.0	5.625 E-05	1.00
FDS-8	3.7500	1.0 E +05	1.0	5.625 E-05	100.00
PDS-1	0.5000	1.0 E -02	0.2520	10.0 E-04	1.0
PDS-2	1.5812	1.0 E -03	0.250	10.0 E-04	1.0
PDS-3	2.5000	1.0 E -04	0.250	2.5 E-04	1.0
PDS-4	3.7500	1.0 E-05	0.250	5.625 E-05	1.0
PDS-5	0.5000	1.0 E-02	0.500	10.0 E-04	1.0
PDS-6	1.5812	1.0 E-03	0.500	10.0 E-04	1.0
PDS-7	2.5000	1.0 E-04	0.500	2.5 E-04	1.0
PDS-8	3.7500	1.0 E-05	0.500	5.625 E-05	1.0
PDS-9	0.5000	1.0 E-02	0.750	10.0 E-04	1.0
PDS-10	1.5812	1.0 E-03	0.750	10.0 E-04	1.0
PDS-11	2.5000	1.0 E-04	0.750	2.5 E-04	1.0
PDS-12	3.7500	1.0 E-05	0.750	5.625 E-05	1.0
PDS-13	0.5000	1.0 E-02	0.250	10.0 E-04	100.0
PDS-14	1.5812	1.0 E-03	0.250	10.0 E-04	100.0

PDS-15	2.5000	1.0 E -04	0.250	2.5 E-04	100.0
PDS-16	3.7500	1.0 E -05	0.250	5.625 E-05	100.0
PDS-17	0.5000	1.0 E -02	0.500	10.0 E-04	100.0
PDS-18	1.5812	1.0 E -03	0.500	10.0 E-04	100.0
PDS-19	2.5000	1.0 E -04	0.500	2.5 E-04	100.0
PDS-20	3.7500	1.0 E -05	0.500	5.625 E-05	100.0
PDS-21	0.5000	1.0 E -02	0.750	10.0 E-04	100.0
PDS-22	1.5812	1.0 E -03	0.750	10.0 E-04	100.0
PDS-23	2.5000	1.0 E -04	0.750	2.5 E-04	100.0
PDS-24	3.7500	1.0 E -05	0.750	5.625 E-05	100.0

Conversion factors:

$$1 \text{ Ft} = 0.305 \text{ Mt}$$
$$1 \text{ Ft}^2/\text{Sec} = 0.0929 \text{ Mt}^2/\text{Sec.}$$
$$1 \text{ Ft}^3/\text{Sec} = 0.028 \text{ Mt}^3/\text{Sec.}$$

to 2.0. Thus the early time drawdown in a well with storage is influenced by the transmissibility of the aquifer.

5.5.3 Effects of partial screening

a) Time domain

It is observed that the merging time is different for different screening ratios. However, for a particular screening ratio, it is found that the product of u_m, (the non-dimensional time parameter corresponding to the merging time, t_m) and the storage parameter, β, remains constant. Also, it is observed that this product increases with increase in the storativity ratio, thus reducing the value of merging time. Further, it is noticed that as the ratio r/L increases the product $u_m \cdot \beta$ also increases. These values, i.e., the product $u_m \cdot \beta$ for different screening ratios, at different values of r/L ratios in each case of storativity ratio is presented in Table 3 . From the close observation of the values for different r/L ratios (in the Table 3) for a particular screening ratio and for a given storativity ratio, it can be seen that as the radial distance from the well increases for a particular leakage factor, the product of $u_m \cdot \beta$ is increasing indicating that t_m is decreasing. That is, as r increases, the different curves corresponding to different βs merge at an early time. It is observed that the merging time is far low in the case of leaky aquifers. This signifies that in the case of leaky aquifers, the different values for β_s merge with the 'no storage' curve at an early time.

TABLE - 3 NONDIMENSIONAL MERGING TIME

SL. NO.	Screening ratio	$u_m \cdot \beta$ for $S'/S = 1.0$				$u_m \cdot \beta$ for $S'/S = 100.0$				c_m for $r/L=6.25 \times 10^{-4}$	
		$r/L=6.25 \times 10^{-4}$	0.01	0.03	0.10	$r/L=6.25 \times 10^{-4}$	0.01	0.03	0.10	$S'/S=1.0$	$S'/S=100$
1.	1.0	9.0×10^{-3}	1.30	14.0	105.0	13.0×10^{-3}	2.40	24.0	200.0	27.78	19.23
2.	0.75	8.2×10^{-3}	1.20	12.0	70.0	9.4×10^{-3}	1.90	20.0	130.0	30.49	26.60
3.	0.50	5.0×10^{-3}	1.10	10.0	63.0	6.0×10^{-3}	1.60	16.0	120.0	50.00	41.67
4.	0.25	4.0×10^{-3}	0.84	7.0	44.0	4.3×10^{-3}	1.00	8.0	90.0	62.50	58.14

The coefficient c_d in the expression for the time of deviation, t_d is a constant with its value as $c_d = 0.125$ ($u_{wd} \cdot \beta = 2.0$)

b) Space domain

As in the case of 'wells with no storage', the partial screening causes increased drawdowns for a particular discharge. Also, it is found from the computer output that the effect of partial screening is felt only up to a radial distance of 1.5 times the thickness of the aquifer, as in the case of confined flow with 'no well storage'.

5.5.4 Mergind distance

As in the case of fully screened wells, the merging distance, i.e., the radial distance where the type curves for different storage parameters will merge approximately (within a reasonable percentage of error, in this case it is assumed as 5%) with the type curve for the 'wells with no storage', is found to be at r/L ratio of 1.0. Also, it is found that this distance reduces with increase in the storativity ratio.

In conclusion, it can be said that the type curves presented herein for various screening ratios are,, in general, similar to the ones presented for fully screened wells. The merging time increases with decrease in the screening ratio. However, the deviation time remains constant for all the screening ratios and storativity ratios for which the analysis is made.

6.0 CONCLUSIONS

In the present investigation, the storage effects for large diameter wells (both fully and partially screened wells) in leaky aquifers were studied. Because of the complexity of the problem, the finite element method based on variational formulation of the problem has been employed, and flow domain around the wells has been discretised as an assemblage of network of rectangular elements. Finer grid of rectangular elements nearer to the well and a Course network of elements farther from the well with triangular elements in the transition zone is used in the model. Variation of the hydrological potential in each element has been assumed to be linear. Crank-Nicolson scheme is chosen for the time integration. The algorithm is so developed that the non-zero upper diagonal terms of conductance and porosity matrices can be stored in rectangular array to save the computer memory requirements. The Gaussian elimination technique is used to solve the resulting simultaneous equations. The finite element models developed for the present study have been verified with the standard known solutions, like, unsteady radial flow towards wells in confined aquifers (Hantush solution), unsteady flow in a leaky aquifers (Hantush solution) and unsteady flow towards well with storage in confined aquifers (Papadopulos and Cooper solutions).

Type curves have been presented for fully and partially screened large diameter wells with storage in leaky aquifers. It has been observed from the type curves that the effects of well storage reduces both with time and increase in radial distance from the well but increase with the storage parameter β . The time from which the storage effects become negligible at a given radial distance from the pumping well is designated as merging time t_m. The concept of time of deviation, t_d, is introduced and a method for the assessment of transmissivity of aquifer from early time pumping history using the deviation time is presented.

The analysis indicated that the effects of leakage through aquitard and storage effects are cumulative and hence steady state conditions are attained much earlier in leaky aquifers for the case of wells with storage compared with the case of wells with no storage. Also it has been found that the effects of leakage increase with increase in storativity ratio between the aquitard and aquifer while the effect of partial screening of well increases the drawdowns.

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