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A Finite Element Model for River Flow Simulation

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ABSTRACT

A finite element model with options of simulating the kinematics, diffusive, and complete hydrodynamic behaviours of river flow has been developed. The computer program is written in FORTRAN 77 and the program is portable and can be run on both microcomputers and mainframe computers. The model performance with all the three options has been presented for channels of irregular cross-sections. A test application of the finite element hydrodynamic model to the river Jamuna in Bangladesh has been made. The results of test application, in terms of water level and discharge have been compared with the results of a well calibrated modelling system, the MIKE 11 and found very close and satisfactory.

INTRODUCTION

Mathematical modelling of river flow has become an accepted powerful engineering tool these days. Finite difference methods have been extensively used for river flow simulation and currently a good number of software are available commercially. But the finite element method is the new comer in river flow simulation, although this method has been used largely in sub-surface flow simulation. In the present study, considering some advantages, the finite element method coupled with the Galerkin's weighted residual principle is applied to solve the one dimensional spatially varied unsteady flow for predicting the discharge, depth, and velocity in rivers. The model developed has options to simulate the kinematics, diffusive, and complete hydrodynamic behaviours of rivers.

The model performance with all the three options has been presented for channels of irregular cross-sections. A test application of the finite element hydrodynamic model to the river Jamuna, one the largest river in Bangladesh has been made. The results of the test application have been compared with the results of the MIKE 11 modelling system(DHI, 1989), developed by the Danish Hydraulic Institute, Denmark.

THEORETICAL CONSIDERATIONS

Governing Equations

The treatment of unsteady flow in an open channel is credited to Barre'de Saint Venant, a French mathematician who developed the complete one-dimensional equations of unsteady flow on the following assumptions:

(i) The flow is one dimensional i.e. the velocity is uniform over the cross-section and the water level across the section is horizontal.

(ii) The stream line curvature is small and the vertical acceleration is negligible, hence the pressure is hydrostatic.

(iii) The effects of boundary and turbulence can be accounted for through resistance laws analogous to those used for steady state flow.

(iv) The average channel bed slope is small so that the cosine of the angle it makes with the horizontal line may be replaced by unity.

In terms of depth of flow, y, and discharge, Q, the Saint Venant equations may be written as:

$$\frac{\partial Q}{\partial x}+b\frac{\partial y}{\partial t}=q \qquad (1)$$

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\alpha\frac{Q^2}{A}\right)+gA\frac{\partial y}{\partial x}+gQ\frac{|Q|}{C^2AR}=0 \qquad (2)$$

In terms of depth of flow, y, and velocity of flow, Q, these equations may be written as:

$$\frac{\partial y}{\partial t}+y\frac{\partial v}{\partial x}+v\frac{\partial y}{\partial x}-q(x,t)=0 \qquad (3)$$

$$\frac{\partial v}{\partial t}+v\frac{\partial y}{\partial x}+\frac{v}{y}q(x,t)+g\frac{\partial y}{\partial x}+g(S_f-S_0)=0 \qquad (4)$$

where, A= area of flow, C= Chezy resistance coefficient, g= acceleration of gravity, y= stage above horizontal level, Q=discharge, α = momentum distribution coefficient, R= hydraulic radius, q= later inflow, and v= velocity of flow.

Boundary Conditions

Upstream stage or discharge hydrographs are always required for natural rivers and can usually be obtained easily. Since the real stage and discharge relationship is multivalued in unsteady flow, the results along the downstream reaches of the model will be distorted if a steady flow (single-valued) rating curve is applied at downstream limit. Therefore, it may be advantageous to use water stage hydrograph at downstream boundary. However, while water stage hydrographs may be available for some past floods, and can be used for model calibration, they will not be known a prior for the exploitation runs unless the models ends at a lake or reservoir, or tidal condition.

Forms of Models

The solution of the complete one-dimensional unsteady flow equations often results in enormous computer time and storage particularly for floods of long durations. This has attracted significant interest in the use of simplified models, such as the kinematic and diffusion flow models. If the momentum equation is rearranged with the friction slope, S_f being the subject of the formula and letting q(x,t) equal zero, the resulting equation is:

$$S_f = S_o - \frac{\partial y}{\partial x} - \frac{v}{g}\frac{\partial v}{\partial x} - \frac{1}{g}\frac{\partial v}{\partial t}$$

Steady uniform flow
(Kinematic model)

Steady non-uniform flow
(Diffusive model)

Steady non-uniform flow

Unsteady non-uniform flow
(Dynamic model)

In rivers with sufficiently steep slope and without backwater effects the terms can be neglected as being small compared with the bed slope. The resulting relationship is known as the kinematic model, and the momentum equation is expressed simply as:

$$S_f = S_0 = \frac{n_1^2 v^2}{R^{\frac{4}{3}}} = \frac{v^2}{R^{\frac{4}{3}} M^2} \quad \text{for Manning's equation and}$$

$$S_f = S_0 = \frac{v^2}{C^2 R} \quad \text{for Chezy's equation}$$

where n_l is the Manning's roughness coefficient and M is the Strickler coefficient.

The kinematic model has been successfully applied in simulating flows in natural floods in steep river slopes of the order of 2m per km or more, overland flows, and slow rising hydrographs. The solution of kinematic equation requires one initial value of dependent variable at each points and one boundary condition. Dropping inertia terms and differentiating with respect to x and t the momentum equation may be written as parabolic partial differential convection-diffusion equation which is a good model for flood propagation if the inertia terms are in fact negligible. It is capable of presenting the backwater influence of tributaries, dams, etc., since it requires two boundary conditions one upstream and one downstream, as in the case for any diffusion equation.

FINITE ELEMENT FORMULATION

Figure 1 shows a mesh for the finite element method. Here, the solution is approximated by a piecewise continuous function such as a series of straight line.

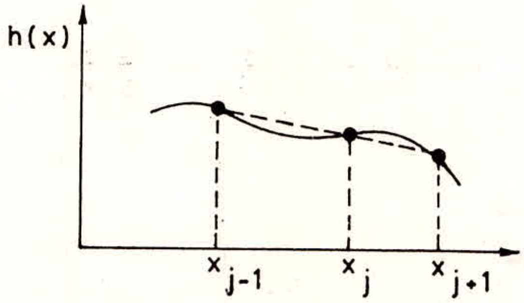

Figure 1. Finite Element Computational Mesh

The piecewise continuous function can be expressed as:

$$h(x,t)=\sum_{j=1}^{N}\phi_j(x)\,h_j(t) \quad \ldots\ldots (6)$$

where the $\phi_j(x)$ are interpolating functions, $h_j(t)$ are a time varying approximations of the exact continuous solution at mesh point j, and N is the total number of nodes.

After the piecewise continuous approximation has been written, it must be adjusted to give a best fit to the exact continuous solution. Two techniques are commonly used to perform the adjustment, the method of weighted residual and an approach based on the calculus of variation.

When the piecewise continuous approximation is substituted into the partial differential equation, a residual will exist, because, the exact continuous solution is not matched. The residual may be defined as:

$$R=\langle L(h)\rangle-L(h)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where L (h) is the partial differential equation and <L(h)> is the discretized form of the partial differential equation. To obtain a 'best fit' between the piecewise continuous approximations and the exact continuous solution, the residual R must be minimized. In the method of weighted residuals, the residual R is multiplied by a series of weighing functions, W_i, integrated over the length of the solution domain L and set to zero:

$$\int_L W_i R dx=0, i=1,\ldots\ldots\ldots N\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

In principles, there are now N equations for N unknown solution variables at the mesh points.

Implementation of the finite element formulation of the flow equations is carried out in four basic steps: (1) channel discretization and selection of approximation function, (2) derivation of element equations, (3) assembly of element equations, and (4) transient solutions of the systems of equations. The details of finite element formulation is shown elsewhere (Hoque, 1989).

The natural channel shown in Figures 2(a) and 2(b) is idealized as an straight line as presented in Figure 2(c); because the flow equations are one-dimensional. The channel is divided into small reaches called elements. Each element will be modeled with the same flow equations but with different channel geometry and hydraulic parameters. The element equations are later assembled into global matrix equations for solution. By applying the Galerkin's principle(Hoque, 1989; Hoque, 1988; Segerlind, 1976) to the continuity equation the following equation is obtained:

$$\sum_{i=1}^{k-1}\int_{x_k}^{x_{k+1}} N^T\left(\frac{\partial y}{\partial t}+y\frac{\partial v}{\partial x}+v\frac{\partial y}{\partial x}-q(x,t)\right)dx=0\ldots\ldots\ldots\ldots(9)$$

in which $\sum_{i=1}^{k-1}$ is the expression for summary individual element equations from 1 to (k-1) elements and N^T transpose to the shape functions.

Using the shape functions, Equations (9) may be written as:

$$\sum_{i=1}^{k-1}\int_0^1 N^T\left[\frac{\partial y}{\partial t}+Y\frac{\partial v}{\partial x}+v\frac{\partial y}{\partial x}-q(x,t)\right]Lds=0\ldots\ldots\ldots\ldots(10)$$

In matrix form the global continuity equation can be written as:

$$[A]\frac{dy}{dt}+[B]y-\{C\}=0 \quad (11)$$

where A,B are the matrices and C is the column vector; y is the dependant variable. The global momentum equation can be formed similarly.

The solution of time dependant global matrix Equation (11) is sought through a semi-discrete approach. This approach requires the time derivative of the dependant variable at each node to be replaced by finite difference scheme in time domain. The forward, backward, and central difference schemes are given below with time level k as:

Forward difference,

$$\frac{dy}{dt}=\frac{y^{k+1}-y^{k}}{\Delta t} \quad (12)$$

Backward difference,

$$\frac{dy}{dt}=\frac{y^{k}-y^{k-1}}{\Delta t} \quad (13)$$

Central difference,

$$\frac{dy}{dt}=\frac{y^{k+1}-y^{k-1}}{2\Delta T} \quad (14)$$

Substitution of Equation (12) in Equation (11) yields

$$[A]\frac{y^{k+1}-y^{k}}{\Delta t}+[B]y^{k}-\{C\}=0 \quad (15)$$

An implicit equation will be generated from Equation (15) with the aid of the time weighing factor.

The kinematic model is developed in implicit form as a set of non-linear tridiagonal matrix equations which are solved by the Newton-Rapson iterative method. The diffusion and the complete hydrodynamic models result in non-linear bitridiagonal matrix equations which are solved by the triangular decomposition technique(Von Rosenberg, 1969). Formulation of the numerical models to predict depth, velocity, and discharge is given elsewhere (Hoque, 1989).

COMPUTER IMPLEMENTATION

A computer program has been written to implement the finite element solution algorithms developed for solution of unsteady flow equations in open channel in different forms such as kinematic wave flow, diffusive wave flow, and dynamic flow as described in the preceding sections. The program has been written in FORTRAN IV language and run on an IBM PC/AT microcomputer. The computer program is simple and portable and with minor changes can be run on any mainframe computers and on the microcomputers with at least 2 MB random access memory (RAM). The program has three options either to implement kinematic flow model, or diffusion flow model, or the complete hydrodynamic model according to the need of the users. The capability of the model to simulate the in bank river flow with each of three options have been verified.

MODEL VERIFICATION AND TEST APPLICATION

The accuracy of the numerical solutions of the unsteady flow equation in open channel can be evaluated by analyzing the river flow problems for which analytical solutions are available, the model results can then be compared with the analytical results. The validity of numerical model can also be evaluated by comparing its performance with the performance of the existing numerical models for the same problem.

Application to Irregular Channel

The channel discretization is shown in Figure 3. Initial depths of flow were generated by backwater calculation starting from a downstream depth. Discharge values at intermediate nodes were estimated by linear interpolation applied to the two initial discharges at upstream and downstream locations. Nodal velocities corresponding to initial depths are calculated by dividing the nodal discharge by corresponding cross section. At the upstream point, the discharge was prescribed as a function of time. At the downstream boundary rating curve was imposed. The initial discharge values are given by the unsteady nonuniform flow of 18 m3 /sec at point A and 22 m3 /sec at point B (see Figure 3) at time t = 0. Figures 4 and 5 show the observed discharge hydrographs at station A and B and the rating curves at these Stations. Computed flow at Station B, 81 km from station A was compared to the observed at the same Station.

The kinematic, diffusive, and the complete flow models are run for a simulation period of 96 hours. The results in terms of discharge, depth, and velocity for every time step at every node are generated. The results in terms of discharge, depth, and velocity for nodes 1, 14, and 26 are plotted in Figures 6, 7, and 8.

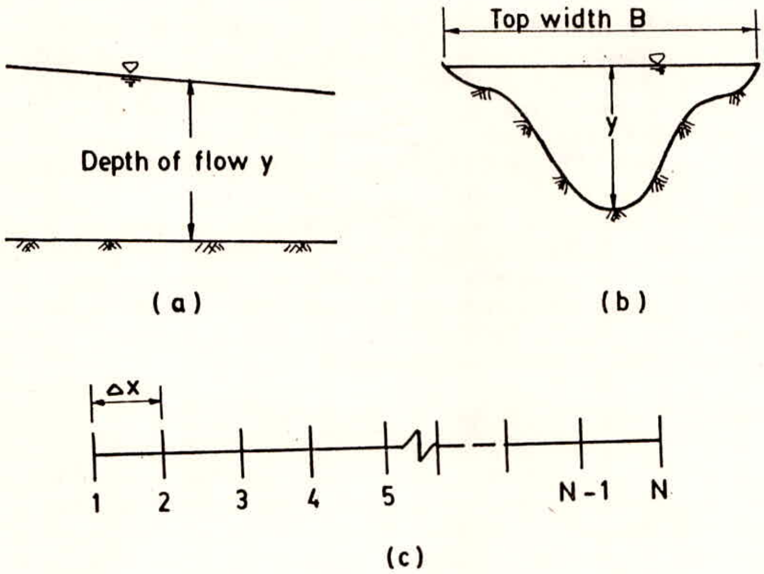

Figure 2. Natural Idealized Sections: (a) Longitudinal Profile, (b) Vertical Cross-Sections, (c) Channel Discretized into Finite Elements.

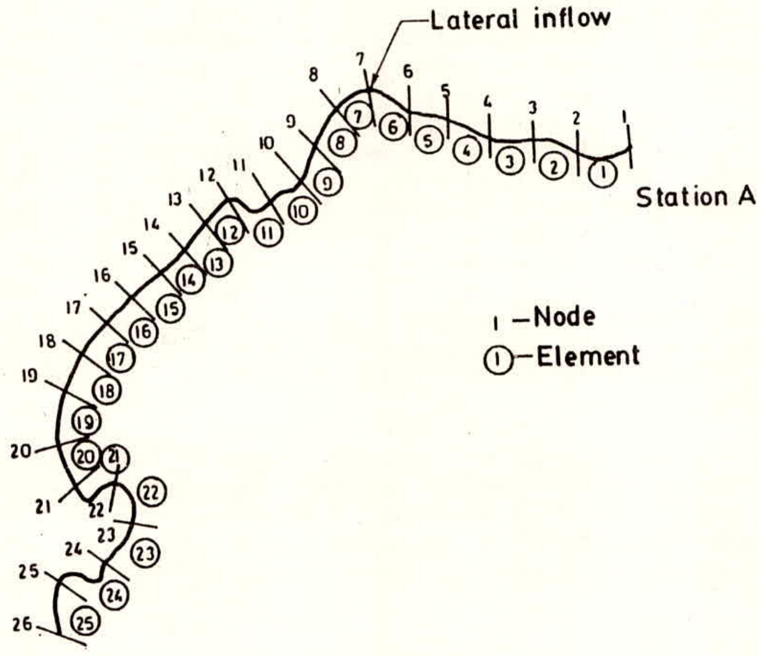

Figure 3. Discretization of an Irregular Channel.

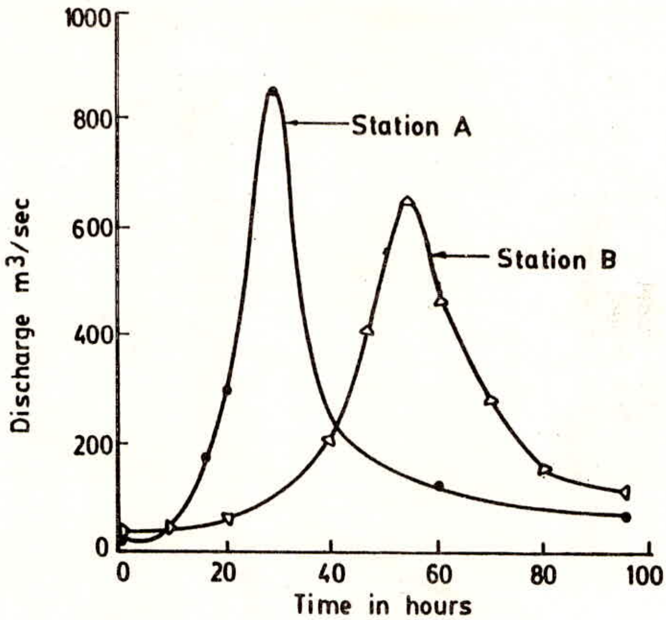

Figure 4. Discharge Hydrograph at Boundaries of Irregular Channel AB.

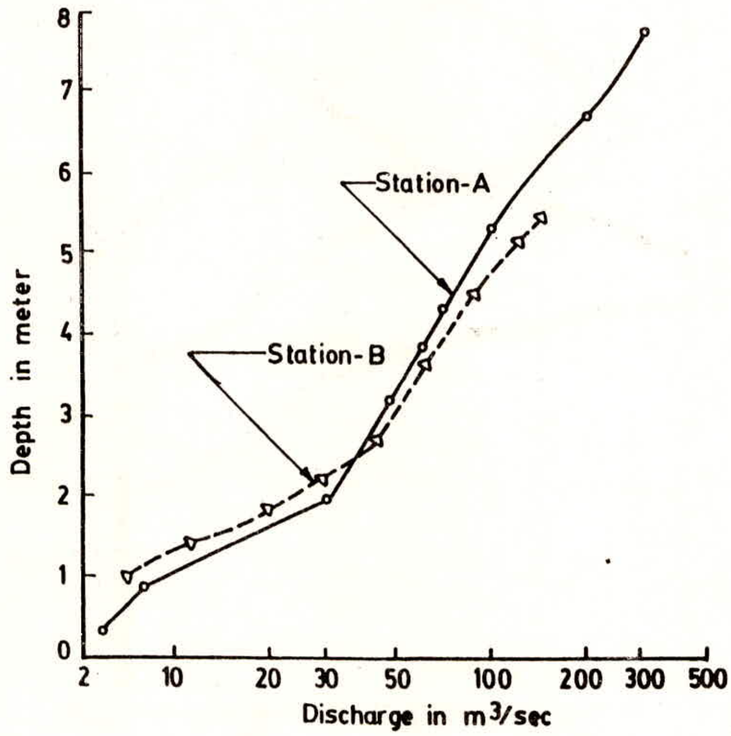

Figure 5. Rating Curve at Boundaries of Irregular Channel AB.

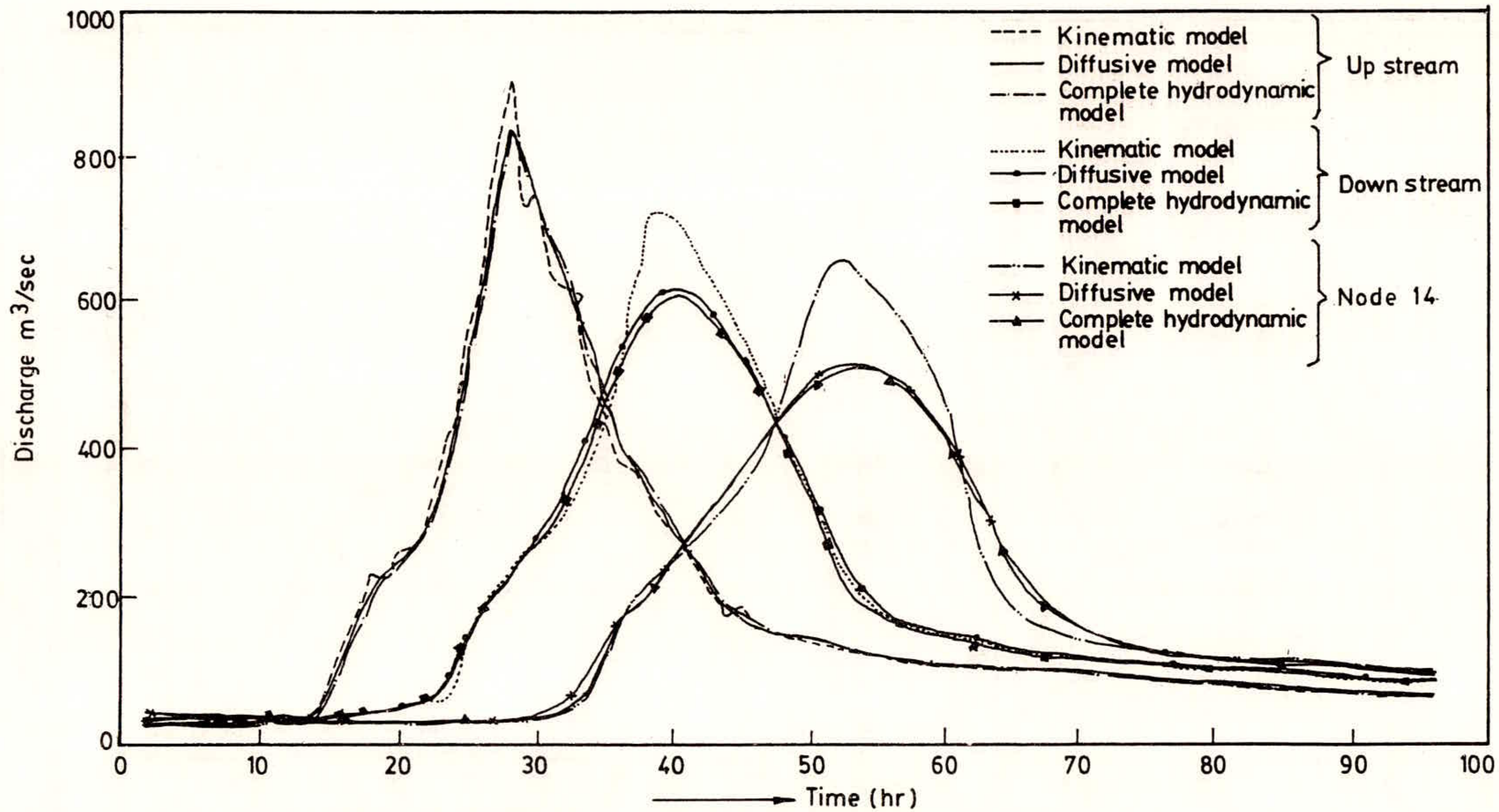

Figure 6. Comparison of Discharge Hydrographs

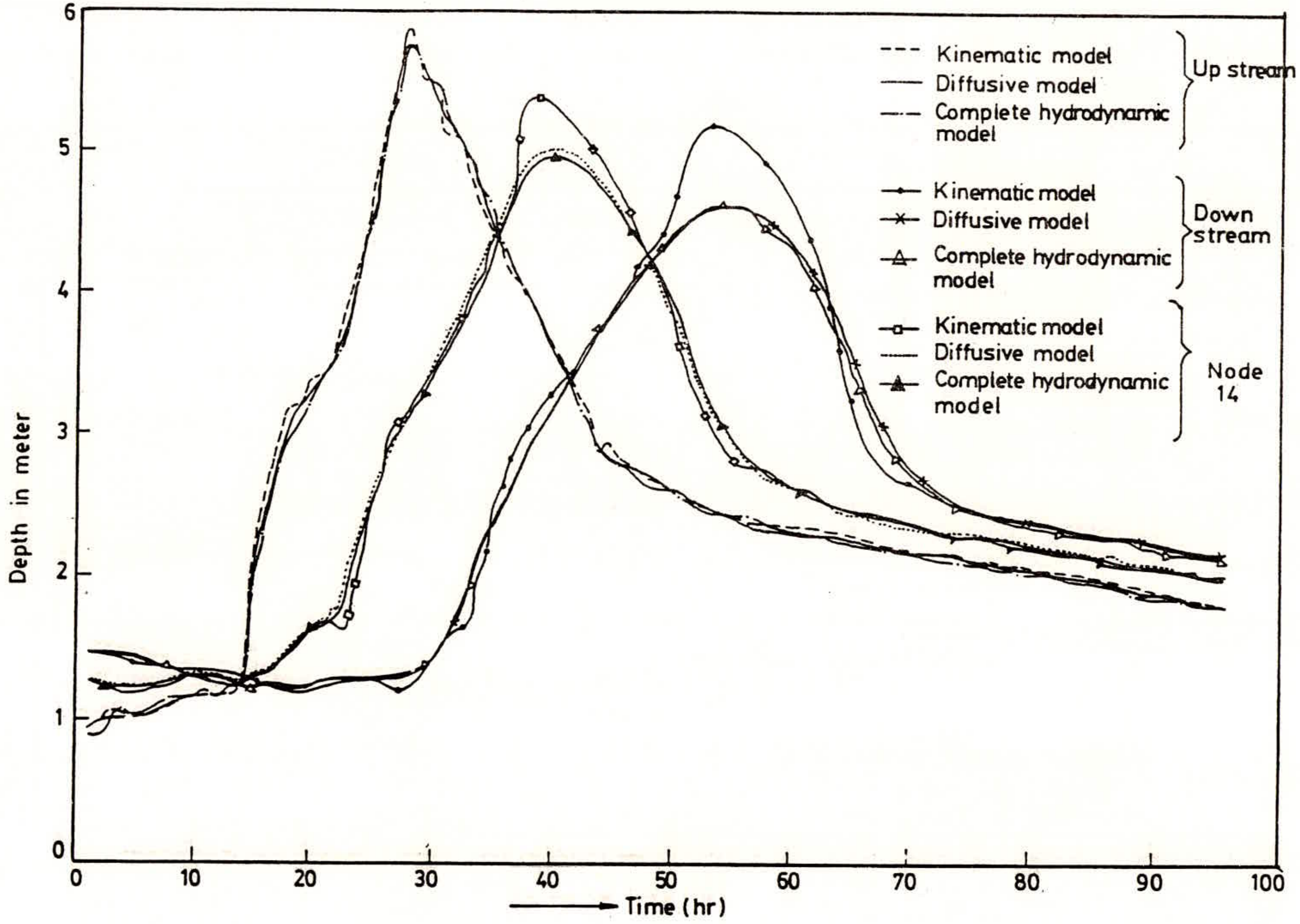

Figure 7. Comparison of Water level Hydrographs

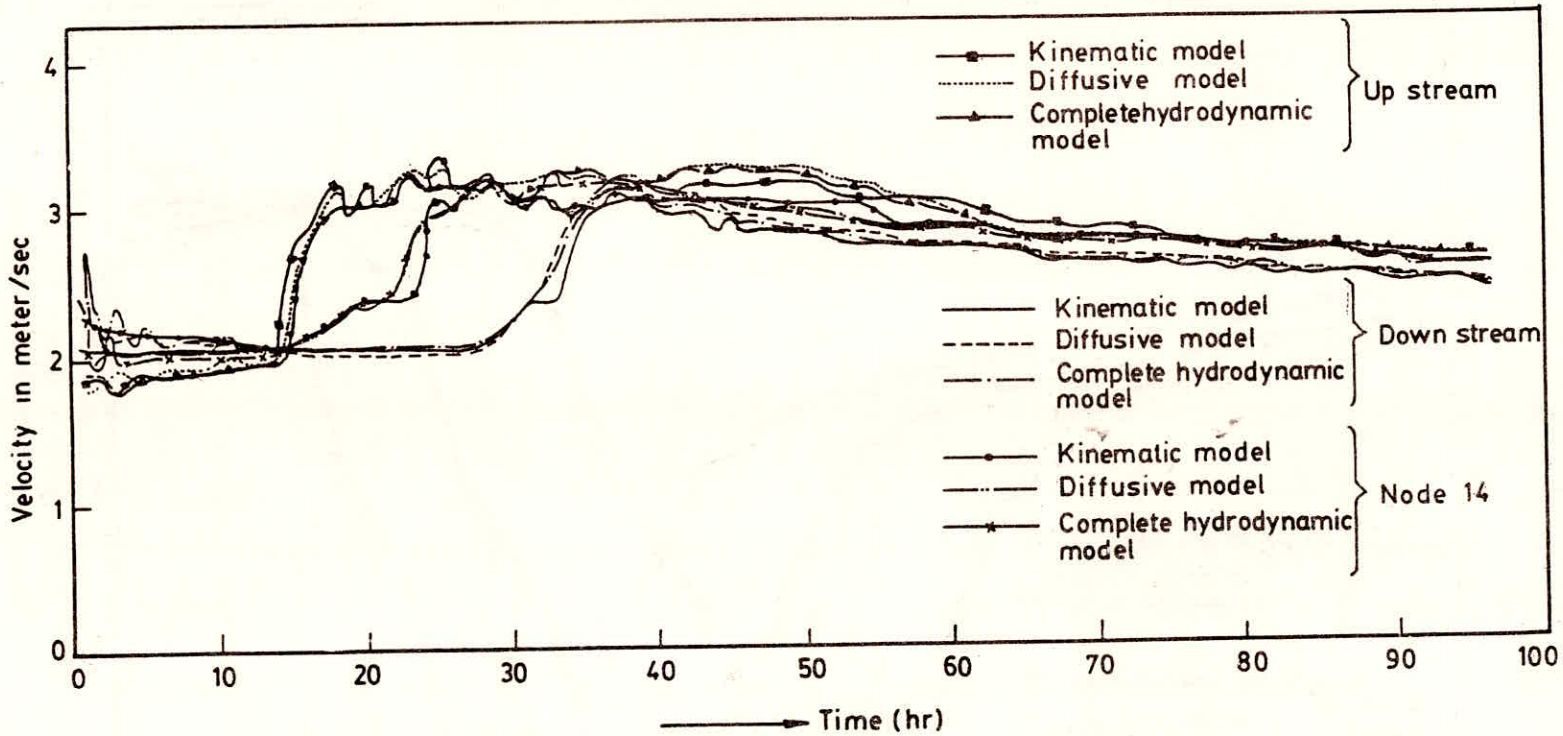

Figure 8. Comparison of Velocities

Application to Natural Big River

The river Jamuna is selected for test application of the model in a natural big river. The mean annual flood peak of the river Jamuna is 65,200 m3/sec. The river reach of about 230 km from Noonkhawa to Aricha as shown in Figure 9(a) is considered. The model schematization is shown in Figure 9(b). The reach is divided into 17 elements on the basis of the availability of cross-sectional data. The discharge hydrograph at Noonkhawa and the water level hydrograph at Aricha are considered as upstream and downstream boundaries, respectively. The internal inflow and outflow between Noonkhawa and Aricha are not taken into account for this test application. In addition to the finite element hydrodynamic model, the hydrodynamic component of the MIKE 11 (DHI, 1989) modelling system of Danish Hydraulic Institute, Denmark is applied to compare the results of the two models which is the main objective of the test application. The MIKE 11 is a well known river modelling system and currently work is under progress at Bangladesh Water Development Board for adaptation of MIKE 11 for flood forecasting in the major rivers of Bangladesh. The MIKE 11 hydrodynamic component is developed using the implicit finite difference technique.

The finite element hydrodynamic model and the hydrodynamic component of the MIKE 11 have been run for a period of two months, April-May, 1986. The results obtained by the models in terms of water level and discharge hydrographs at 40 km downstream from Noonkhawa are plotted with time in Figures 10(a) and 10(b), respectively. It is observed that the finite element model has produced very close results to that of the MIKE 11, although some phase differences are observed. However, it may be concluded that the finite element hydrodynamic model developed is capable of simulating flow in large natural rivers.

CONCLUSIONS

The results presented have proved the ability of the finite element model to produce good approximate solutions to a variety of river flow problems. Thus, from this study the following conclusions may be made:

1. For management of surface water this model may be used as a basic tool to evaluate the effects of varying withdrawal rates and pattern for planning purpose.

2. This model can be coupled with a groundwater flow model and used to evaluate surface water and groundwater interactions.

3. This model can be used as a component of water resources forecasting management model for formulating a plan for conjunctive use of surface water and groundwater.

4. This model can be coupled with a transport diffusion model and used to analyze movements of pollutants and salinity in rivers.

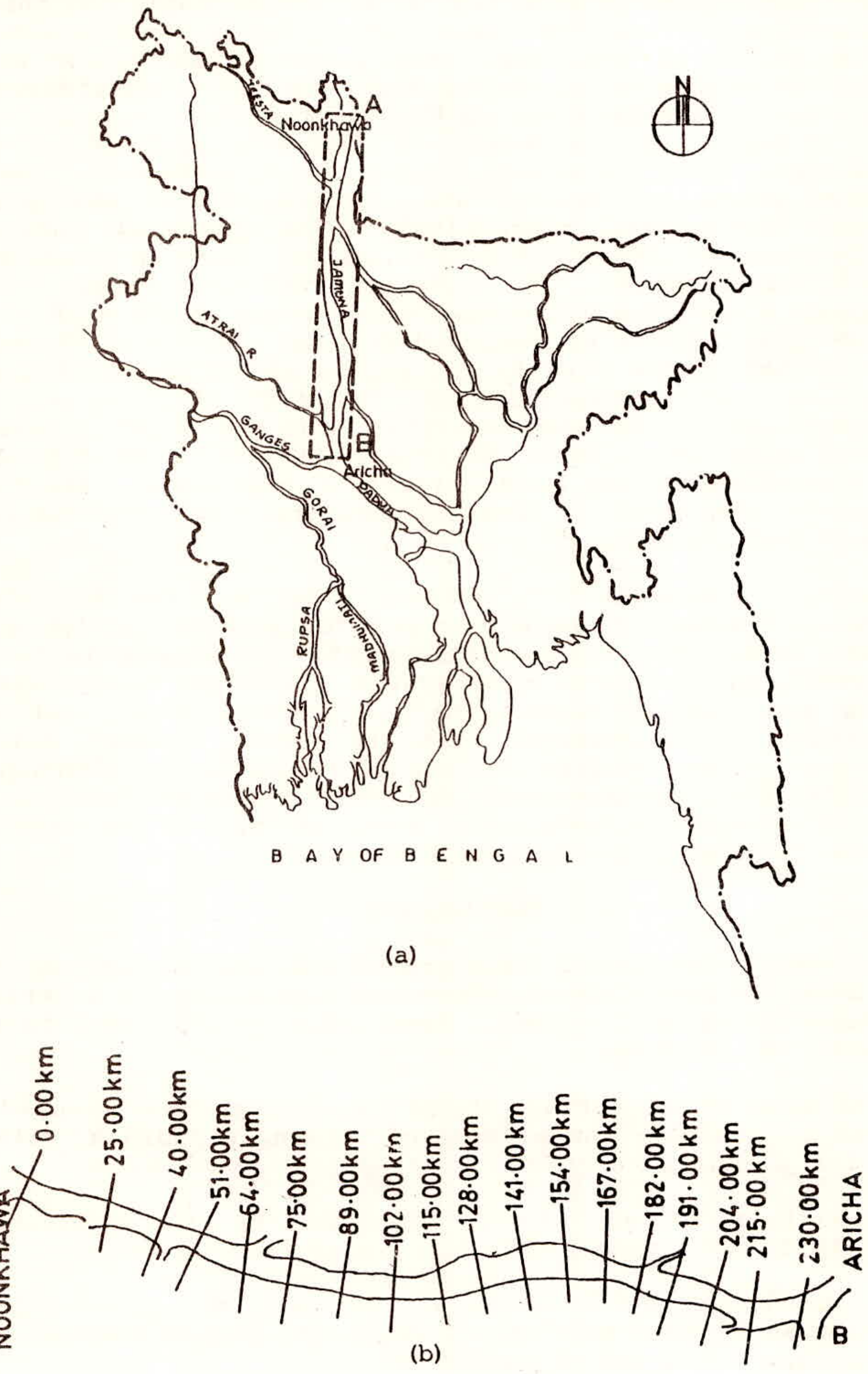

Figure 9. Model Schematization: (a) Bangladesh Map Showing Modelled Reach AB of the River Jamuna, (b) Model Discretization.

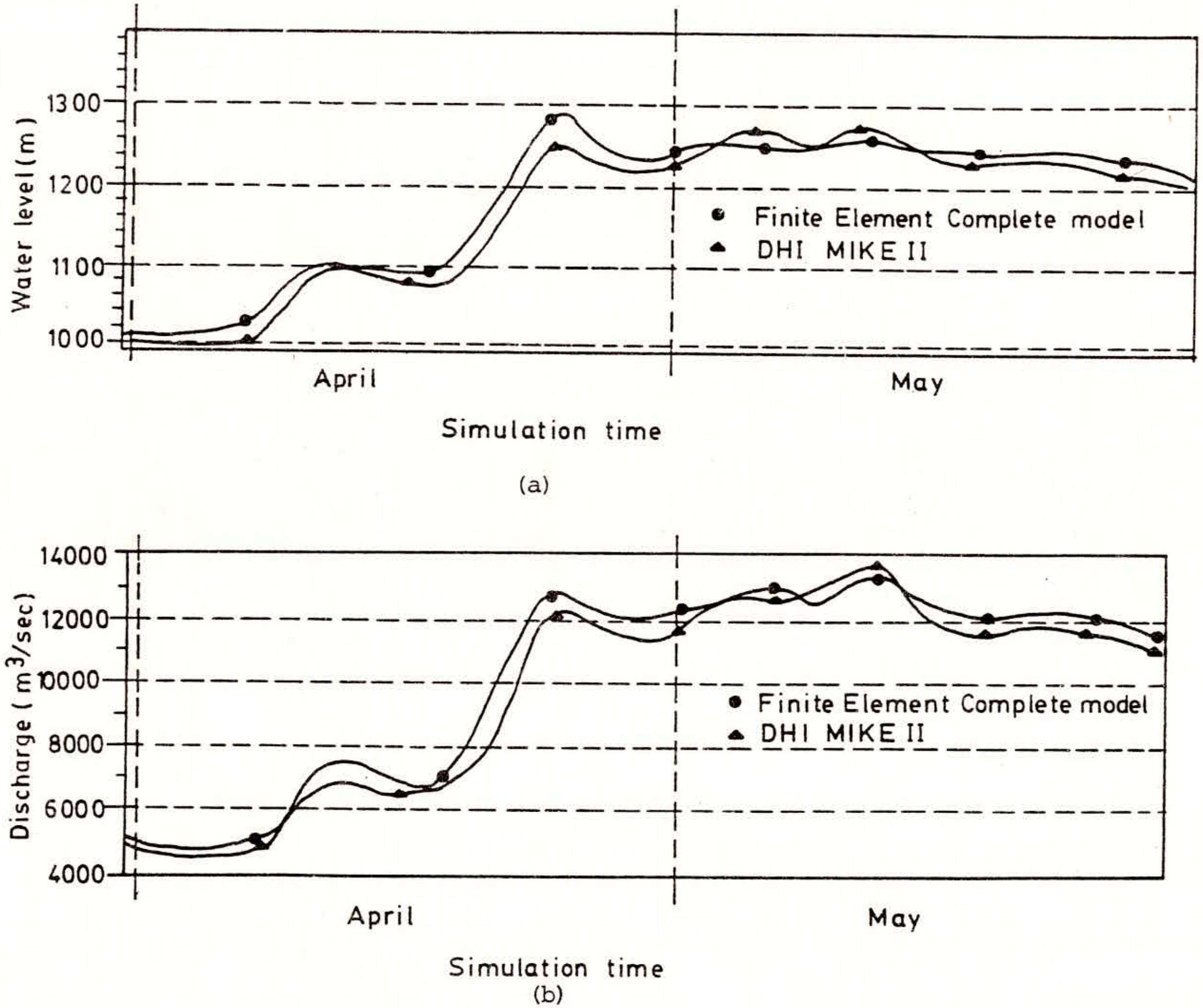

Figure 10. Comparison of Results: (a) Water level Hydrograph, (b) Discharge Hydrograph.

ACKNOWLEDGEMENT

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Comments by the author on the observations by General Reporter

1. The finite element method has some inherent advantages over the finite difference method, particularly in handling the more complex network and boundary conditions. The present paper is, however, based on application of finite element method in simulating flow in a single channel, where hardly any advantage of finite element over finite difference could be shown. When the present model is extended to a network of channels, the advantages of finite element over finite difference would be presented.

2. During monsoon season the Jamuna river, in many places, flows overbank. The model presented can not handle the overbank flow or flood plain. Therefore the application of the model could not be extended to a complete year.

3. The watershed does not belong to a mountainous one.

4. A regular time step of 1 hour has been used in simulation.