INTRODUCTION TO LINEAR PROGRAMMING

by

Dr. S K Jain,
Scientist, National Institute of Hydrology, ROORKEE.

INTRODUCTION

Optimization is the science of choosing the best amongst a number of possible alternatives. For many engineering problems, a number of feasible solutions are available. It is required to evaluate each alternative and then choose the best from the point of view of interest, say economic or convenience etc. The best solution under the given circumstances is known as the **optimum** solution.

The variables which are manipulated to obtain the optimum solution are termed as decision variables. The decision maker's preference is shown by the objective function. The availability of resources is usually limited and is expressed with the help of constraints. The optimization techniques (also known as mathematical programming techniques) can be classified in several ways. One useful way of classifying the techniques is based on the nature of the problem itself. Following this way the techniques can be classified as Linear Programming, Non-linear Programming, and Dynamic Programming.

LINEAR PROGRAMMING

The optimization problems in which the objective function and constraints are linear function of decision variables along with the condition that the decision variables are positive are termed as Linear Programming (LP) problems. Although the objective function and decision variables are not linear in nature in many water resources problems, they can be approximated by linear functions and LP can be used to obtain the solution.

The standard and expanded form of LP problem is :

$$\text{Min } Z = \Sigma\, c_i x_i \qquad (1)$$

Subject to

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \;....\; + a_{1n}x_n \ge b_1\\
&a_{22}x_1 + a_{22}x_2 + \;....\; + a_{2n}x_n \ge b_n\\
&\;\;\cdot \qquad\quad \cdot \qquad\qquad\qquad \cdot \qquad \cdot\\
&a_{m1}x_1 + a_{m2}x_2 + \;....\; + a_{mn}x_n \ge b_m\\
&\qquad x_i \ge 0,\; i=1,2,....n.
\end{aligned} \qquad (2)$$

Here, x_i are the decision variables. c_i are the cost coefficients (or benefit coeff.) and represent the cost incurred by increasing the x_i decision variable by one unit. The right hand side of constraint equations represents resource availability. These arise due to limited availability of a particular resource, say water. The a_{ij} coefficients are called technological coefficients and quantify the amount of a particular resource i required per unit of the activity j.

If the number of decision variables n is equal to the number of constraint equations m, the problem has a unique solution, if it exists. If m > n, and if m-n equations are not redundant, it has a solution only in least square sense.

If m < n, then we can set (n-m) variables equal to zero and solve the m equations for m variables. However, there will be ${}^{n}C_{m}$ such solutions. Each of these solutions is called a basic solution. The (n-m) variables which have been set equal to zero are called non-basic variables; remaining n variables are called basic variables. A basic solution which satisfies all the constraints is called a basic feasible solution and any such solution which provides minimum (or maximum) value of the objective function is called an optimum solution. The feasible region and constraints are shown in Fig. 1.

A constraint of ≥ type can be easily converted to a ≤ type by multiplying by -1 throughout the equation. An inequality constraint of ≥ type can be converted to equality type by introducing a variable $s_i \geq 0$. Thus the constraint :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1 \quad (3)$$

is equivalent to :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n - s_1 = b_1 \quad (4)$$

The variable s_1 is called surplus variable.

Similarly, an inequality constraint of the type ≤ can be converted to equality type by introducing a slack variable s_i. Hence the constraint :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \quad (5)$$

can be written as :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 = b_1 \quad (6)$$

MATRIX REPRESENTATION OF A LP PROBLEM

In matrix notation, a LP problem may be presented as :

$$\text{Min } Z = C^T X \quad (7)$$

subject to :

$$A X \geq b \quad (8)$$

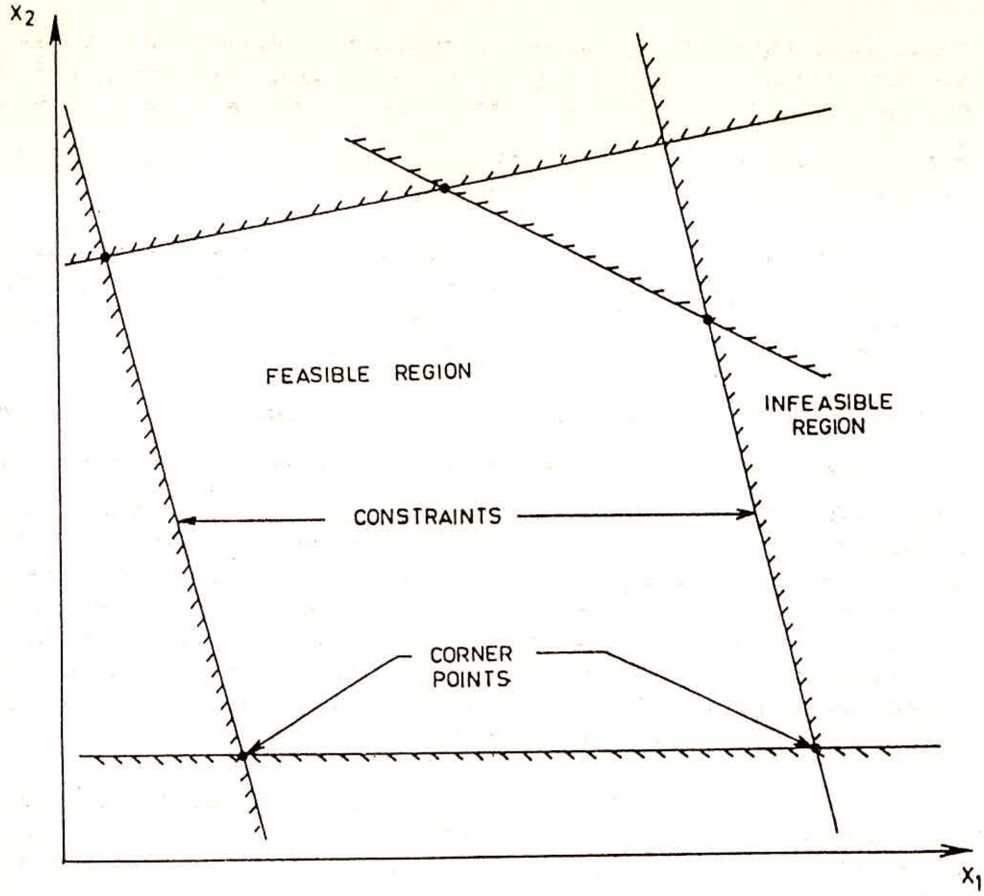

FIGURE -1 FEASIBLE REGION AND CONSTRAINTS

$$X \geq 0$$

where,

$$X = [x_1 \; x_2 \ldots\ldots x_n]$$

$$C = [c_1 \; c_2 \; \ldots\ldots c_n]^T$$

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

STANDARD FORM OF A LP PROBLEM

A LP problem is written in standard format as :

$$\text{Min } Z = c_i \; x_i \qquad (9)$$

subject to :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \; \ldots . + a_{1n}x_n &= b_1 \\ a_{22}x_1 + a_{22}x_2 + \; \ldots . + a_{2n}x_n &= b_n \\ \vdots \qquad \vdots \qquad & \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \; \ldots . + a_{mn}x_n &= b_m \end{aligned} \qquad (10)$$

and $x_i \geq 0, \; i=1,2,\ldots.n.$

Suppose in a particular problem, n = 20, and m = 10, then the number of possible basic solutions will be $^{20}C_{10}$ or

$20!/[(20-10)! \; 10!] = 184756.$

Hence to solve this problem, 184756 solutions will be required to be obtained and compared. This is formidable task even with the help of a fast digital computer. A very efficient method was developed by Dantzig which is called Simplex Method. Before discussing the Simplex Method, graphical solution of a LP problem is being discussed.

GRAPHICAL SOLUTION OF A LP PROBLEM

Here a LP problem in two dimensions will be discussed.

$$\text{Max } Z = 2 \; x_1 + x_2$$

subject to :

$$2x_1 - x_2 \le 8$$
$$x_1 + x_2 \le 10$$
$$x_2 \le 7$$
$$x_1, x_2 \ge 0$$

In the Fig. 2, the constraints are plotted against the coordinate axes x_1 and x_2. The non-negativity constraints are plotted as the axes themselves. To mark the constraint $2x_1 - x_2 < 8$, we plot a straight line $2x_1 - x_2 = 8$. Similarly, we plot lines $x_1 + x_2 = 10$, and $x_2 = 7$ to mark second and third constraints. The feasible region can be easily delineated and is shown by hatched lines. Now we start with a particular value of objective function, say 6 and plot the line $2x_1 + x_2 = 6$. Since it is a maximization problem, the objective function line is shifted forward as far as possible while ensuring that at least one point lies in the feasible region. It can be seen that the farthest point up to which we can go is the point (6,4). Hence, this is the optimum point at which the objective function is equal to 16 and $x_1 = 6$ and $x_2 = 4$.

A closer inspection of Fig. 2 will show that the optimum point will always be a corner point.

SIMPLEX METHOD

The graphical method of solving a LP problem is good only for problems involving two decision variables. As the size of the problem increases, this method cannot be used. Dantzig invented an efficient method, named Simplex method, of solving LP problems. This method is discussed here briefly.

To begin with, we first transform the problem into canonical form. The characteristics of canonical form are :

(a) The basic variables have positive unit coefficients and only one of them (different each time) appears in each equation,

(b) The basic variable do not appear in the objective function,

(c) The RHS of constraints must be positive.

One way is to arbitrarily choose the basic variables and use a technique like Gauss Elimination to transform the equations in the canonical form. If the equations contain only the slack variables, these can be automatically considered as basic variables. But in case problem has surplus variables and equality constraints, we introduce artificial variables in the equation. An auxiliary objective function is formed which is equal to the sum of artificial variables. The computations are demonstrated using the same example as taken above.

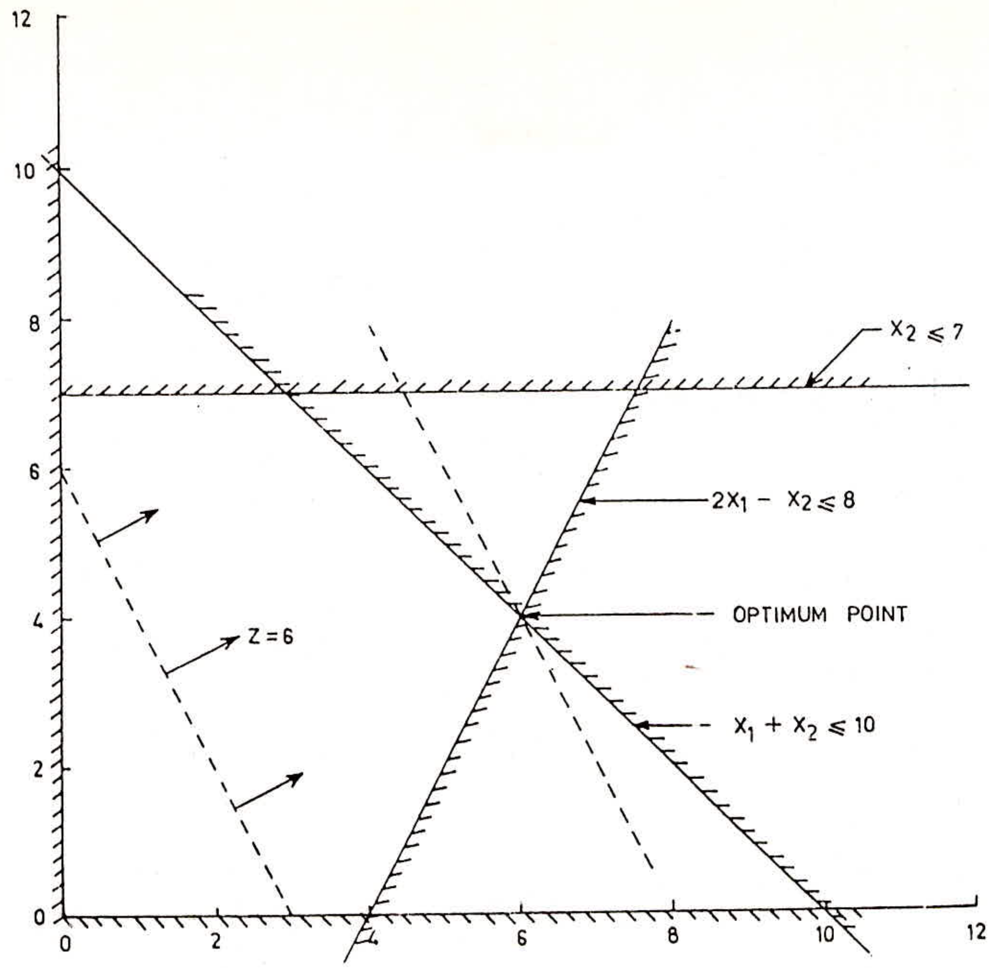

FIGURE -2 GRAPHICAL SOLUTION OF A LP PROBLEM

$$\text{Max } Z = 2x_1 + x_2$$

subject to :

$$2x_1 - x_2 \leq 8$$
$$x_1 + x_2 \leq 10$$
$$x_2 \leq 7$$
$$x_1, x_2 \geq 0$$

Writing this problem in standard form by introducing surplus variables:

$$\text{Max } Z = 2x_1 + x_2$$

subject to :

$$2x_1 - x_2 + x_3 = 8$$
$$x_1 + x_2 + x_4 = 10$$
$$x_2 + x_5 = 7$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Now in this problem, the number of variables n is 5 and number of constraints m = 3. Hence, there will be two basic variables which could be chosen arbitrarily. If the coefficients of x_1 and x_2 were +1, the equations would have been in the canonical form. This not being the case, we divide the first constraint equation by 2. The problem now looks like :

$$\text{Max } Z = 2x_1 + x_2$$

subject to :

$$x_1 - 0.5x_2 + 0.5x_3 = 4$$
$$x_1 + x_2 + x_4 = 10$$
$$x_2 + x_5 = 7$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The problem is now in the canonical form.

This method is called two phase simplex as we have two objective functions. The first phase aims at minimization of the auxiliary objective function. If as a result of this phase, this function can not be made zero then the problem is infeasible and the algorithm is terminated. If W = 0, then the optimization of the main function is taken up.

For ease of computations, the Simplex Tableau is formed as follows :

RHS	x_1	x_2	x_3	x_4	x_5
4	1	-0.5	0.5	0	0
10	1	1	0	1	0
7	0	1	0	0	1
-Z=-O	2	1	0	0	C
4	1	-0.5	0.5	0	0
6	0	1.5	-0.5	1	0
7	0	1	0	0	1
-Z=-8	0	2	-1	0	0
6	1	0	0.25	1	0
4	0	1	-1/3	2/3	0
1	0	-0.5	0.5	-1	1
-Z=-16	0	0	-1/3	-4/3	0

COMPUTATIONAL STEPS OF SIMPLEX METHOD

(i) After Simplex Tableau is formed, it is checked whether the objective function will improve by replacing a basic variable. A solution will be optimal if all the cost coefficients are positive or zero in a minimization problem or are negative or zero in a maximization problem.

If optimality is not satisfied then the variable which will improve the objective function at the fastest rate, i.e., for which cost coefficient is most negative (for minimization) or most positive (for maximization) is brought in the basis. The decision is arbitrary in case of a tie. Let this variable be x_r.

(ii) Now for this variable, we take b_i/a_{ir} ratio for each constraint row i (with +ve a_{ir}) and the minimum ratio determines the row in which the basic variable will have unit coefficient. The corresponding variable from this row (which was a basic variable) will leave basis. The equations are again converted into canonical form by suitable row and column operations. These steps are repeated until an optimal solution is found.

Nowadays very efficient computer packages are available for solution of LP problems which make the use of this technique very attractive. Post optimality analysis in an important part of any optimization problem. This is concerned with finding out as to how the optimal solution changes with change in the various coefficients of a problem. Other topics of interest in linear programming include the transportation problem and the assignment problem. Reference is made to Rao(1979) and Taha(1976) for further reading.

REFERENCES

Rao, S.S., Optimization, Theory and Practice, Wiley Eastern, 1979.

Taha, H., Operations Research, an Introduction, Macmillan Publishing Company, 1976.

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