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UNSTEADY FLOW TO A WELL IN MULTIPLE CONFINED AQUIFERS

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ABSTRACT

Transient ground water flow to a fully penetrating well installed in n-confined aquifers (n is an integer) separated by aquicludes has been analysed. The aquifers were in hydraulic equilibrium before the start of pumping. Explicit equations for drawdown distribution in each aquifer as well as its contribution to the well discharge have been developed. The solution for drawdown distribution in each aquifer is simple like Theis equation in which aquifer transmissivity is replaced by an effective transient transmissivity of the aquifer. When the diffusivities of the aquifers are equal, the solution reduces to Theis equation. The solution is approximate and is valid for large times such that the variable $r_w^2 S_i/4T_i t < 0.05$. The analysis has revealed that (i) the contribution of an aquifer to the well discharge during transient state is governed by the aquifer diffusivity, (ii) at the start of pumping the ground water flow to the well is more from an aquifer of the least aquifer diffusivity, and (iii) the flow from aquifers of equal diffusivities is invariant over time and is proportional to aquifer transmissivities with proportionality constant being equal to the inverse of the sum of the aquifer transmissivities. The difference in drawdowns between two radial distances in each aquifer follow a Thiem type equation.

INTRODUCTION

Very often wells draw water from a complex multiple aquifer system. Such a system consists of a series of aquifers separated from each other by confining layers of relatively low hydraulic conductivity. These confining beds may be considered aquitards or aquicludes depends upon whether the leakage through them is appreciable or not. Hydraulics of wells in nonleaky and leaky aquifers have primarily been developed for wells having their screens in a single aquifer (Hantush, 1964). Exception to this tend of analysis has been the work of Sokol(1963) who was the first to derive the steady state solution to hydraulic head changes in aquifers connected by a non-pumping well. Papadopulos (1966) obtained asymptotic solutions for transient discharge and hydraulic head distributions around a well tapping two-confined aquifers of different piezometric levels when the well was initially unpumped and then pumped with a constant rate. The duration for

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which his asymptotic solution is valid is not specified. Khader and Veerankutty (1975) analysed unsteady ground water flow to a well having screens in two aquifers - a water table aquifer overlying a confined aquifer. They employed Laplace and Hankel transforms in conjunction with Schapery's method of inversion to develop drawdown and discharge relationships in each aquifer for a well of (i) zero discharge and different initial heads, and (ii) constant discharge and identical initial heads.

Shakya and Singh (1986) and Shakya et al. (1986) developed analytic solutions to several problems of steady ground water recharge through a well having screens in two leaky confined aquifers. Equations for rate of recharge and hydraulic head distributions in each leaky aquifer were developed for a constant hydraulic head (Shakya and Singh, 1986) and constant injection (Shakya et. al., 1986) boundary conditions at the well bore. The latter also incorporated the effect of accretion from a ponded land surface. Using discrete kernal approach Mishra et. al. (1985) analysed unsteady flow to a well in two aquifers separated by an aquiclude. They made extensive computations and found that the contribution of an aquifer to the well discharge is governed by its hydraulic diffusivity value. Nautiyal (1984) followed the same method of analysis to solve problems of transient ground water flow to a well installed in (i) more than two aquifers separated by aquicludes, and (ii) two aquifers separated by an aquitard. Though the discrete Kernel approach is elegant it requires successive computation to get the contribution of each aquifer to the well discharge as well as drawdown at a particular time.

The objective of this study was to develop analytic solutions for unsteady ground water flow to a well installed in n-confined aquifers separated by aquicludes, where n is an integer. The initial piezometric levels of all the n-aquifers were assumed to be the same and the well screens fully penetrated each aquifers. Equations using exponential integrals have been obtained for (i) the drawdown distribution in each aquifer, and (ii) contribution of each aquifer to the well discharge computations of drawdown and discharge for a well in two confined aquifers have been compared with analytic-numeric results of Mishra et al.(1985).

THEORETICAL ANALYSIS

Consider a constant discharge well fully penetrating n-confined aquifers, each separated by an aquiclude. The aquifers are homogeneous, isotropic and infinite in areal extent with transmissivities T_i, and storage coefficients S_i, $i=1,2,\ldots n$. Before the start of pumping the piezometric heads in all the aquifers are in equilibrium (Fig.1).

The following form of the differential equation in cylindrical coordinates adequately describes the flow of ground water to the well in each aquifer

$$\frac{\partial^2 s_i}{\partial r^2} + \frac{1}{r}\frac{\partial s_i}{\partial r} = \frac{S_i}{T_i}\frac{\partial s_i}{\partial t} \tag{1}$$

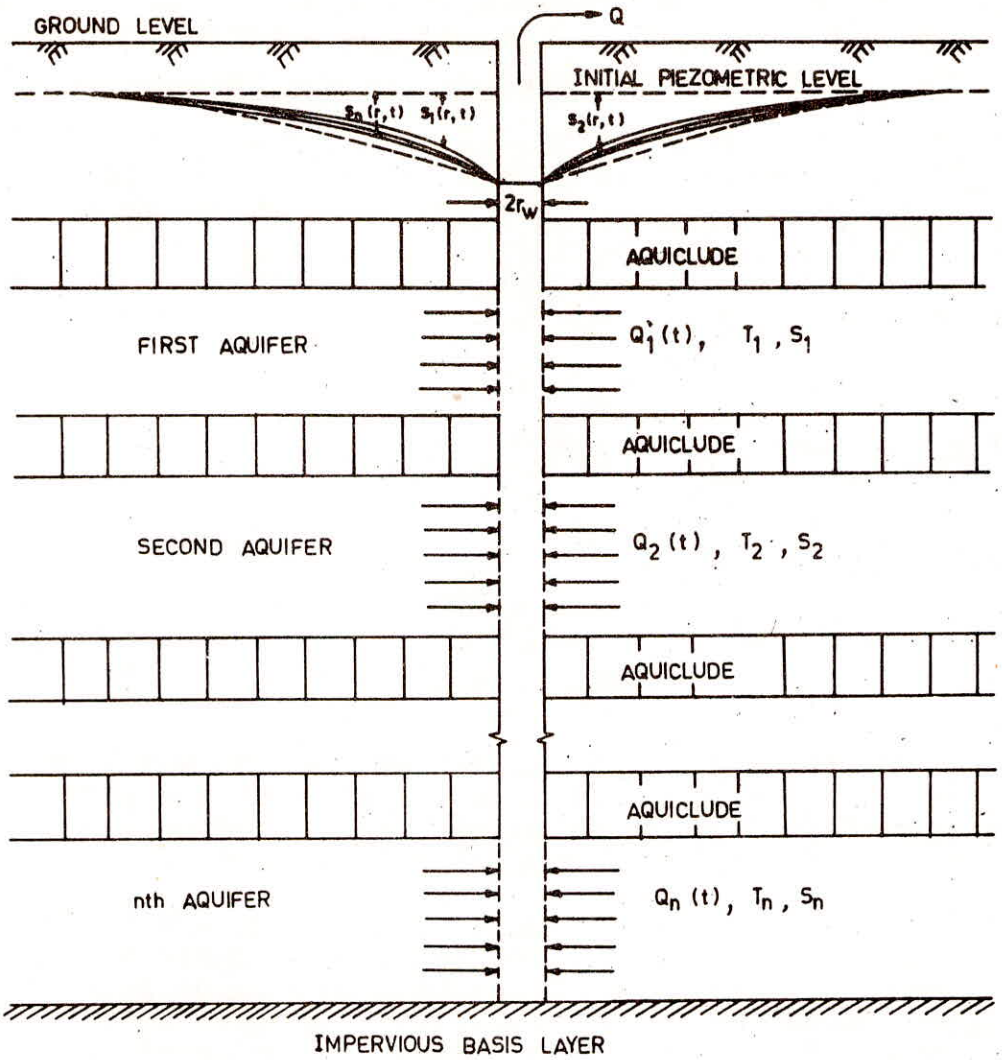

FIG. 1—A FULLY PENETRATING WELL IN n-CONFINED AQUIFERS

for $i = 1,2,\ldots,n$,

$r_w \le r < \infty$,

in which $s_i(r,t)$ is the drawdown in the ith aquifer at radial distance r and time t, $T_i = k_i b_i$ is the aquifer transmissivity L^2T^{-1}, $S_i = S_{s_i} b_i$ is the storage coefficient, dimensionless, K_i is the hydraulic conductivity, LT^{-1}, S_{s_i} is the specific storativity, L^{-1}, b_i is thickness of the aquifer, L, r_w is the radius of the well, L, and subscript i to any quantity is for the ith aquifer. The auxiliary conditions to (1) are

$$s_i(r,0) = 0 \text{ for } r > r_w \tag{2a}$$

$$\lim_{r \to \infty} s_i(r,t) = 0 \tag{2b}$$

$$\lim_{r \to r_w} 2\pi r \sum_{i=1}^{n} T_i \frac{\partial s_i}{\partial r} = - Q \tag{2c}$$

and

$$\lim_{r \to r_w} s_1(r,t) = \lim_{r \to r_w} s_2(r,t) = \ldots$$

$$= \lim_{r \to r_w} s_n(r,t) \tag{2d}$$

The differential equations (1) in conjunction with the auxiliary conditions (2a,b,c,d) constitute a system of n simultaneous partial differential equations which are coupled only at the well bore through the boundary conditions (2c) and (2d).

The usual approach for solving the boundary value problem(1) and (2) is with the help of the integral transform techniques. Very often the transforms of the boundary value problems have been approximated (Hantush, 1964, Papadopulos, 1966, Khader and Veerankutty, 1975) to develop solutions for short and large times. Here an approach is devised in which in place of approximating the transform, the solution itself is assumed and later on its validity is established. In well hydraulics the approximation is introduced at the level of differential equation itself. The differential equation of flow of slightly compressible fluids in deformable porous media is nonlinear (Bear, 1972). It is approximately linearised by neglecting the nonlinear term resulting in (1). The error on account of the neglect of the nonlinear term has been analysed by Singh and Sagar (1980) for flow of water and petroleum in deformable media.

The solution of ground water flow to a fully penetrating well in a confined aquifer involves exponential integral(Theis, 1935). Large time solution of the constant head well problem is also approximated by the exponential integral (Hantush, 1964, p 309). Herein the solution of the boundary value problem (1) and (2) is assumed as the product of the exponential integral, W(u), and an arbitrary function of time, i.e.,

$$s_i(r,t) = A_i(t)\, W(u_i), \quad i = 1,2, \ldots, n, \tag{3}$$

in which

$$W(u_i) = \int_{u_i}^{\infty} \frac{e^{-x}}{x}\, dx, \tag{4}$$

$$u_i = \frac{r^2 S_i}{4T_i t}, \quad i = 1,2,\ldots, n. \tag{5}$$

and $A_i(t)$, $i = 1,2,\ldots,n$, are unknown time dependent arbitrary functions to be determined as part of the solution. Herein, the functions $W(u_i)$ are chosen as they satisfy the initial and boundary conditions (2a) and (2b). The unknown functions $A_i(t)$ are determined to satisfy the boundary conditions (2c) and (2d) at the well bore.

Substituting (3) in (2d) and solving for $A_j(t), j=2,3,\ldots,n$, in terms of unknown $A_1(t)$, one gets

$$A_j(t) = \frac{A_1(t)\, W(u_{1w})}{W(u_{jw})} \tag{6}$$

where

$$u_{jw} = \frac{r_w^2 S_j}{4T_j t} \tag{7}$$

The function $A_1(t)$ is determined by using the boundary information (2c). The derivative of (4) with respect to r could be obtained using chain rule and Leibnitz rule. It is helpful in evaluating the space derivative of the drawdown.

Substituting

$$\frac{\partial s_i}{\partial r} = -2A_i \frac{e^{-u_i}}{r} \tag{8}$$

and the value of A_i from (6) in (2c), one gets

$$A_1(t) = \frac{Q}{4\pi F_1} \tag{9}$$

Putting (9) in (6) yields

$$A_i(t) = \frac{Q}{4\pi F_i} \tag{10}$$

where

$$F_i = W(u_{iw}) \sum_{j=1}^{n} \frac{T_j\, e^{-u_{jw}}}{W(u_{jw})} \tag{11}$$

for $i = 1,2,\ldots,n$.

F_i has the same dimension as aquifer transmissivity. Hence it could be considered as effective transient transmissivity of the ith aquifer in n-aquifer system. The system of equations(3) together with (9) and (10) is the solution to the system of partial differential equations (1).

Quantity of Flow to the Well From Each Aquifer

Let $Q_i(t)$ is the rate of flux to the well from ith confined aquifer. An application of Darcy Law in conjunction with (8) yields

$$Q_i(t) = 4\pi T_i \; A_i \; e^{-u_{iw}}, \quad i = 1,2, \ldots ,n. \tag{12}$$

The pumpage from the well bore is replenished by the ground water fluxes from the individual confined aquifers. Thus

$$Q = \sum_{i=1}^{n} Q_i(t) \tag{13}$$

At any time t, the ratio of discharges from aquifers i and j is given below

$$\frac{Q_i(t)}{Q_j(t)} = \frac{T_i}{T_j} \; \frac{W(u_{jw})}{W(u_{iw})} \; \frac{e^{-u_{iw}}}{e^{-u_{jw}}} \tag{14}$$

When the well is pumped for a very large time the drawdown in each aquifer reaches steady state. Under such condition (14) approaches to a limiting constant ratio. Using L' Hospital's rule (Fulks, 1967, pp 130-133) on the functions inside the square bracket one gets

$$\lim_{t \to \infty} \left\{ \left[\frac{\frac{\partial}{\partial t} W(u_{jw})}{\frac{\partial}{\partial t} W(u_{iw})} \right] \frac{e^{-u_{iw}}}{e^{-u_{jw}}} \right\}$$

$$= \lim_{t \to \infty} \left[\frac{e^{-u_{jw}/t}}{e^{-u_{iw}/t}} \right] \frac{e^{-u_{iw}}}{e^{-u_{jw}}} = 1 \tag{15}$$

Thus

$$\lim_{t \to \infty} \frac{Q_i(t)}{Q_j(t)} = \frac{T_i}{T_j} \tag{16}$$

Substituting (9) and (10) in (12) one could obtain the ratio of ground water withdrawal from the ith aquifer to the total discharge of the well, i.e.

$$\frac{Q_i(t)}{Q} = \frac{1}{\sum_{j=1}^{n} \left(\frac{T_j}{T_i}\right)\left(\frac{e^{-u_{jw}}}{e^{-u_{iw}}}\right)\left(\frac{W(u_{iw})}{W(u_{jw})}\right)} \tag{17}$$

The limiting value of (17) as $t \to \infty$ could be obtained using L'Hospital's rule as in (15).

Hence

$$\lim_{t \to \infty} \frac{Q_i(t)}{Q} = \frac{T_i}{\sum_{j=1}^{n} T_j} \tag{18}$$

At short time the u_{iw} of an aquifer having smallest diffusivity $D_i = T_i/S_i$, approaches to a large value faster than the u_{jw} of the other aquifer. For large value of u_{iw}, i.e. for $u_{iw} \geq 8$, $W(u_{iw}) = 0$ and (17) becomes

$$\frac{Q_i(t)}{Q} = 1 \tag{19}$$

Hence at short time the entire discharge of the well is drawn from the aquifer having the least aquifer diffusivity. For $D_1 = D_2 = \ldots = D_i = \ldots = D_n$, one gets $u_{1w} = u_{2w} = \ldots = u_{iw} = \ldots = u_{nw}$. Substituting these data in (17) one gets a time invarient ground water withdrawal from the ith aquifer:

$$\frac{Q_i(t)}{Q} = \frac{T_i}{\sum_{j=1}^{n} T_j} \tag{20}$$

Thus the effect of equal aquifer diffusivity on the contribution of an aquifer to the total well discharge is the same as that of the steady contribution to the flow.

With the help of extensive computations Mishra et al.(1985) obtained the results of equations (16), (18) and (19). However, they derived (20) algebraically using discrete kernal approach.

Reduction to Theis Equation

The widely used Theis (1935) equation describes drawdown distribution around a fully penetrating constant discharge well in a homogeneous and isotropic confined aquifer of infinite areal extent. The flow under such condition is radial, and the drawdown is independent of depth. Let us study the drawdown in n-aquifers of equal diffusivities. Thus for $T_1/S_1=T_2/S_2=\ldots = T_n/S_n = D$, one gets $u_1=u_2=\ldots=u_n=u$, and $u_{1w}=u_{2w}=\ldots=u_{nw}$ where

$$u = \frac{r^2}{4Dt}, \tag{21a}$$

and $$u_w = \frac{r_w^2}{4Dt} \tag{21b}$$

Hence when all the aquifers have the same diffusivities, (3) with help of (9) and (10) becomes

$$s_i(r,t) = \frac{Q}{4\pi e^{-u_w} \sum_{j=1}^{n} T_j} W(u) \tag{22}$$

Eqn.(22) gives the same drawdown distribution around a finite radius well of negligible storage in all the confined aquifers. Since $\lim r_w \to 0$, $\exp(-u_w) = 1$, therefore (22) reduces to

$$s(r,t) = \frac{Q}{4\pi \sum_{j=1}^{n} T_j} W(u) \tag{23}$$

Thus the well known Theis equation is also applicable to a well installed in n-aquifers provided they have the same aquifer diffusivities. In (23) the subscript i has been dropped as drawdown is the same in all the aquifers.

Reduction to Thiem's Equation

For $u_i < 0.05$, $W(u_i)$ is approximated by (Hantush, 1964, p. 321)

$$W(u_i) = \ln(0.562/u_i) \tag{24}$$

Thus at large times ($u_i < 0.05$) the drawdown at any radial distance is obtained by substituting (24) in (3). The difference in drawdowns at r_w and r in the ith aquifer is given by

$$s_i(r_w,t) - s_i(r,t) = \frac{Q}{4\pi F_i} \ln\left(\frac{r}{r_w}\right)^2 \tag{25}$$

Thus unlike a well in a single aquifer, the difference in drawdowns at two radial distances in a particular aquifer due to pumping a well in n-confined aquifers does not follow Thiem's equation. But considering F_i as an effective transient transmissivity, (25) is a form of Thiem's equation. However, if the aquifer diffusivities were equal, (25) would reduce to Thiem's equation for an well of zero radius in n-aquifers.

The limiting value of F_i at very large time could be evaluated using L'Hospital's rule and equation (15), which is

$$\lim_{t \to \infty} F_i = \sum_{j=1}^{n} T_j \tag{26}$$

Let $\lim_{t \to \infty} A_i(t) = A_s$. With the help of (26), A_s is obtained as below,

$$\lim_{t \to \infty} A_i(t) = A_s = \frac{Q}{4\pi \sum_{j=1}^{n} T_j} \tag{27}$$

When pumping is continued for a very long period such that steady state has reached, (25) with the help of (27) becomes

$$\lim_{t \to \infty} [s_i(r_w,t)-s_i(r,t)] = \frac{Q}{2\pi \sum_{j=1}^{n} T_j} \ln\left(\frac{r}{r_w}\right) \tag{28}$$

for $i = 1,2,\ldots, n$.

which is Thiem's equation for a well in n-aquifers (Shakya, et al. 1986). It predicts the same drawdown in all the aquifers.

Validity of the solution

The system of equations (3) is an approximate solution to the system of parabolic partial differential equations (1). In order to ensure that (3) is a good approximator to the exact solution, it is necessary to know the time for which the differential equations (1) and the initial and boundary conditions (2a,b,c,d) are satisfied.

Satisfying Auxiliary Conditions

Dividing the numerator and denominator of (3) by $W(u_i)$ and substituting the expression for $A_i(t)$, one gets

$$s_i(r,t) = \frac{Q}{4\pi\left[\frac{W(u_{iw})}{W(u_i)}\right]\left[\sum_{j=1}^{n} \frac{T_j e^{-u_{jw}}}{W(u_{jw})}\right]} \tag{29}$$

$\lim_{t \to 0} s_i(r,t)$ could be evaluated using L'Hospital's rule on the quantities inside the square brackets. Thus for $r > r_w$,

$$\lim_{t \to 0} s_i(r,t) = 0, \text{ for } i = 1,2,\ldots, n,$$

which is (2a), as it should be. Checking the satisfaction of the boundary condition (2b) by (3) is straight forward as

$\lim_{r \to \infty} W(u_i) = 0$ results in $\lim_{r \to \infty} s_i(r,t)=0$ for $i=1,2,\ldots, n$.

Substitution of (8), (12), and (13) in the left hand side of the boundary condition (2c) yields

$$\lim_{r \to r_w} 2\pi r \sum_{i=1}^{n} T_i \frac{\partial s_i}{\partial r} = -4\pi \sum_{i=1}^{n} T_i A_i(t) e^{-u_{iw}} = - Q, \quad (30)$$

which is the same as the right hand side of (2c). Checking the boundary condition (2d) against (3) gives

$$A_1(t) W(u_{1w}) = A_2(t)\, W(u_{2w}) = \ldots = A_n(t)\, W(u_{nw})$$

or

$$\frac{Q\, W(u_{1w})}{4\pi\, W(u_{1w}) \sum_{j=1}^{n} \frac{T_j\, e^{-u_{jw}}}{W(u_{jw})}} = \frac{Q\, W(u_{2w})}{4\pi\, W(u_{2w}) \sum_{j=1}^{n} \frac{T_j\, e^{-u_{jw}}}{W(u_{jw})}}$$

$$= \ldots = \frac{Q\, W(u_{nw})}{4\pi\, W(u_{nw}) \sum_{j=1}^{n} \frac{T_j\, e^{-u_{jw}}}{W(u_{jw})}} \quad (31)$$

Eqn.(31) is an identity. Thus all the auxiliary conditions are satisfied.

Satisfying Differential Equations

Substituting (3) in (1), one gets

$$\frac{\partial^2 s_i}{\partial r^2} + \frac{1}{r} \frac{\partial s_i}{\partial r} = \left(\frac{S_i}{T_i t}\right) A_i(t)\, e^{-u_i} \quad (32)$$

and

$$\frac{S_i}{T_i} \frac{\partial s_i}{\partial t} = \left(\frac{S_i}{T_i t}\right)\left[t \frac{\partial A_i}{\partial t} W(u_i)\right] + \left(\frac{S_i}{T_i t}\right) A_i(t) e^{-u_i} \quad (33)$$

If (3) were the exact solution, the differential equations (1) would have been exactly satisfied, but it is not so, as the right hand side of (32) is equal only to the second term in the right hand side of (33). It implies that $s_i(r,t) = A_i(t)\ W(u_i)$ would be an approximator to the true solution if

$$\frac{S_i}{T_i t}\left[t\ \frac{\partial A_i}{\partial t}\ W(u_i)+A_i\ e^{-u_i}\right] \text{ is closer to } \left[\frac{S_i}{T_i t}\ A_i(t)\ e^{-u_i}\right]$$

Subtracting (33) from (32), one gets $\left[-\frac{S_i}{T_i}\ \frac{\partial A_i}{\partial t}\ W(u_i)\right]$

as a measure of error in the solution. The magnitude of this error at large times is evaluated below.

From (3), (9) and (10) one obtain

$$\frac{\partial A_i}{\partial t}\ W(u_i) = -\frac{4\pi}{Q}\ \frac{A_i^2(t)}{t}\ W(u_i)\left[\frac{Q\ e^{-u_{iw}}}{4\pi\ A_i(t)\ W(u_{iw})}\right.$$

$$\left.+\ W(u_{iw})\ \sum_{j=1}^{n}\frac{T_j\ e^{-u_{jw}}\ u_{jw}}{W(u_{jw})} - \sum_{j=1}^{n}\frac{T_j\ e^{-2u_{jw}}}{W^2(u_{jw})}\right] \qquad (34)$$

Using L Hospital's rule one evaluates

$$\lim_{t\to\infty}\left[-\frac{S_i}{T_i}\ \frac{\partial A_i}{\partial t}\ W(u_i)\right] = 0 \qquad (35)$$

Thus at large times error tends to zero and the solution $s_i(r,t) = A_i(t)\ W(u_i)$ approaches exact solution. The magnitude of time after which Eqn.(3) closely approximates the exact solution is estimated by comparing the results from (17) with those of Mishra et al. (1985) and Khader and Veerankutty (1975). Though the solution of Mishra et al. is semianalytical, it has been considered here as an exact one. Eqn. (17) is a dimensionless relationship giving the ratio of ground water withdrawal from the ith aquifer to the total discharge of the well. Using this equation values of $Q_1(t)/Q$ were computed for a well installed in two confined aquifers to study the accuracy of the proposed solution (3). The aquifers parameters chosen were $S_1/S_2 = 100$ for $T_1/T_2 = 0.1$, 1.0, and 10.0 which resulted into $D_1/D_2 = 10^{-3}$, 10^{-2} and 10^{-1}, respectively. The values of $Q_1(t)/Q$ against u_{1w} are plotted by the dotted lines in Fig.2. The solid lines in the figure depict the solution of Mishra et al. (1985). It is evident from the figure that the results derived from the proposed solution are identical to those of

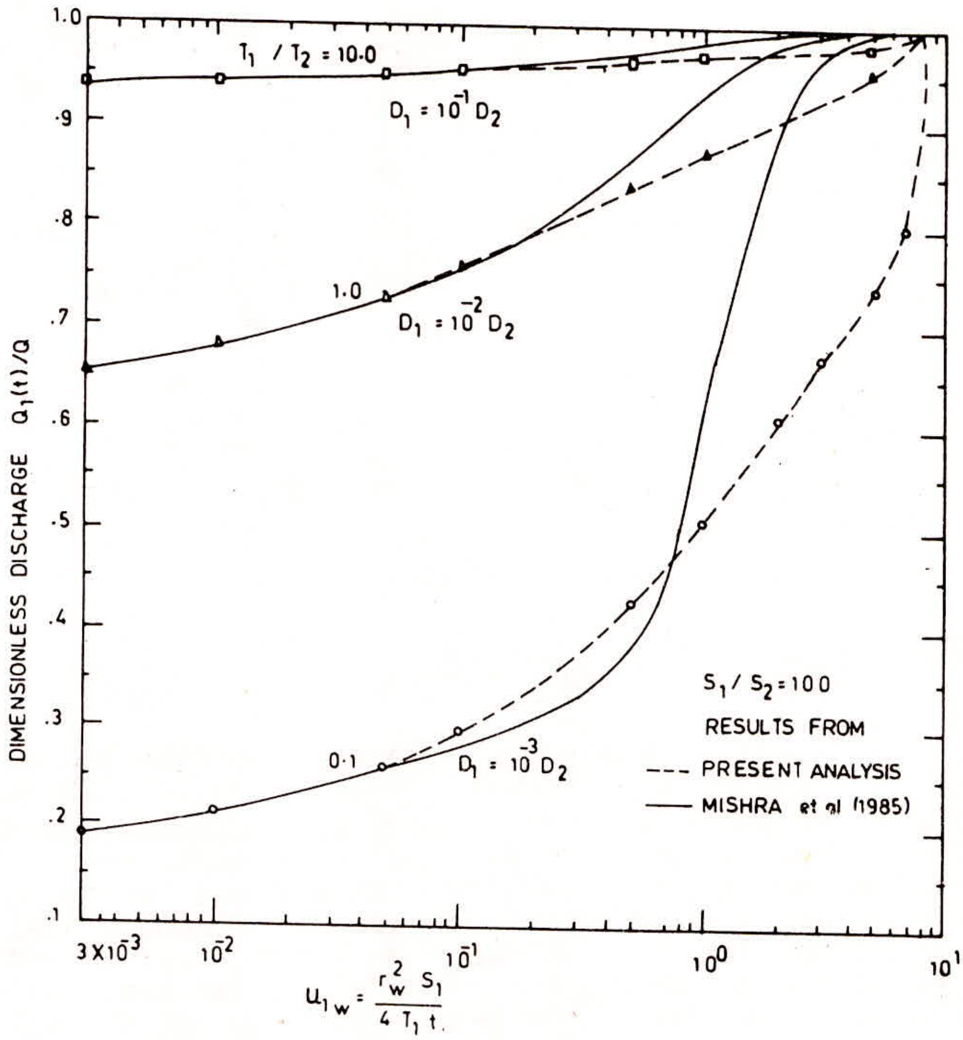

FIG. 2- COMPARISON OF RESULTS FROM THE PRESENT ANALYSIS AND THOSE OF MISHRA et. al. (1985)

of Mishra et al. for all values of $u_{1w} < 0.05$ with increasing differences for larger values of u_{1w}. The two solutions meet again at $u_{1w} = 8$. Since in all the three cases $D_1 < D_2$, therefore $u_{1w} > u_{2w}$ at all times. The wide deviation of the proposed solution (3) from the solution of Mishra et al. for $0.05 < u_{1w} < 8$ is due to the large differences between

$$\frac{S_1}{T_1 t}\left[t \frac{\partial A_1}{\partial t} W(u_{1w}) + A_1(t)\, e^{-u_{1w}}\right]$$

and $\left[\frac{S_1}{T_1 t} A_1(t)\, e^{-u_{1w}}\right]$ in this range.

At early times after the start of pumping the function $A_i(t)$ changes rapidly as shown in Fig.3. This phenomenon is studied by nondimensionalising $A_i(t)$ by dividing it by A_s given by (27). Thus

$$\frac{A_i(t)}{A_s} = \frac{\sum_{j=1}^{n} (T_j/T_i)}{W(u_{iw}) \sum_{u=1}^{n} \frac{(T_j/T_i)\, e^{-u_{jw}}}{W(u_{jw})}} \qquad (36)$$

The dimensionless rate of change in $(A_i(t)/A_s)$ is obtained by differentiating (36) with respect to time and multiplying it again by time.

Hence

$$t \frac{\partial}{\partial t} (A_i(t)/A_s) = - \frac{\sum_{j=1}^{n} (T_j/T_i)}{\left[W(u_{iw}) \sum_{j=1}^{n} \frac{(T_j/T_i)\, e^{-u_{jw}}}{W(u_{jw})}\right]^2}$$

$$\left[e^{-u_{iw}} \sum_{j=1}^{n} \frac{(T_j/T_i)\, e^{-u_{jw}}}{W(u_{jw})} + W(u_{iw})\right.$$

$$\left. \sum_{j=1}^{n} \frac{(T_j/T_i)\, e^{-u_{jw}}\, u_{jw}}{W(u_{jw})} - \sum_{j=1}^{n} \frac{(T_j/T_i)\, e^{-2u_{jw}}}{W(u_{jw})^2}\right] \qquad (37)$$

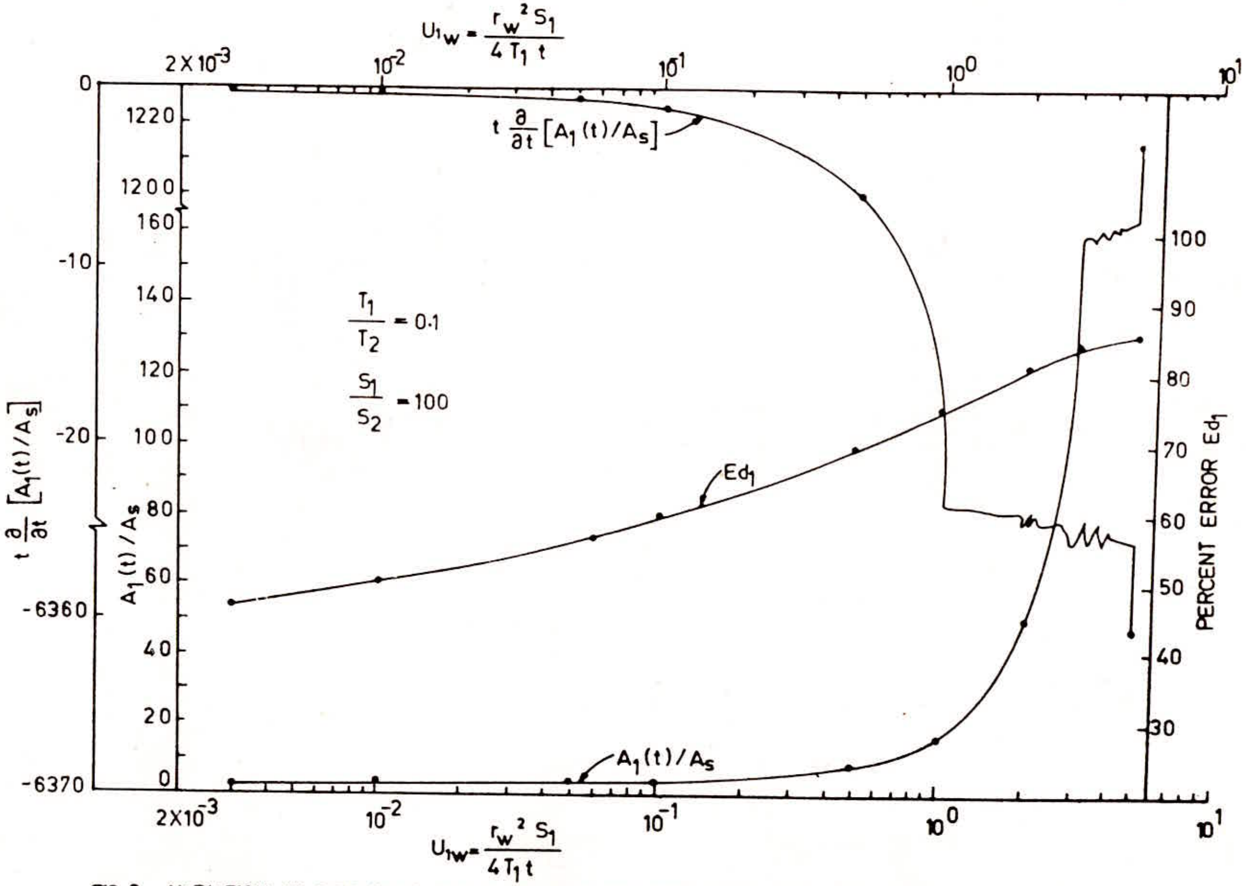

FIG 3 VARIATION IN DIFFERENT TERMS OF THE SOLUTION WITH TIME FOR AQUIFER PARAMETERS $T_1/T_2 = 0.1$, $S_1/S_2 = 100$

Let E_d be an error indicating deviation in the satisfaction of the differential equations by the solutions (3). It is obtained by dividing the error $[-(\frac{S_i}{T_i t}) \quad t \frac{\partial A_i/A_s}{\partial t} W(u_i) \quad]$ by $[(\frac{S_i}{T_i t}) \frac{A_i(t)}{A_s} e^{-u_i}]$ which gives the deviation from unity in the right hand side of (33). The per cent error for the ith differential equation, Ed_i, outside the well screen is defined by

$$Ed_i = - \frac{-100 \; t \frac{\partial}{\partial t}[A_i(t)/A_s] W(u_{iw})}{\frac{A_i(t)}{A_s} e^{-u_{iw}}} \tag{38}$$

Computations of equations (36), (37) and (38) were made for a well screening two confined aquifers. The aquifer parameters were the same as in Fig. 2, i.e. S_1/S_2 = 100 for T_1/T_2=0.1, 1.0, and 10. Plots of $[A_1(t)/A_s]$, $t \frac{\partial}{\partial t} [A_1(t)/A_s]$, and E_{d1} for T_1/T_2 = 0.1, 1.0 and 10.0 are shown in figures 3,4, and 5, respectively. All the three figures depict similar trends. Fig.3 shows a change in $[A_1(t)/A_s]$ from 1214.9487 at u_{1w}=5.0 to 3.0577 at u_{1w}=0.05. The corresponding values of $t\frac{\partial}{\partial t}[A_1(t)/A_s]$ are - 6360.7321 and -0.6663, respectively. The minus sign is due to a decrease in the rate of change $[A_1(t)/A_s]$. In this range of u_{1w} the error Ed_i decreased from 85.47 per cent to 56.54 per cent. Similarly $[A_1(t)/A_s]$ in Fig.4 varied from 279.2563 at u_{1w} = 5.0 to 1.5357 at u_{1w} = 0.05 and in Fig.5 from 160.4158 at u_{1w} = 5.0 to 1.0965 at u_{1w} = 0.05. The trends in $t\partial[A_1(t)/A_s]/\partial t$ and E_{d1} in figures 4 and 5 are also similar to those in Fig.3, but with different magnitudes.

The values of E_{d1} at u_{1w} = 0.05 for T_1/T_2 = 0.10, 1.0, and 10 were computed to be 56.54, 26.51 and 14.72 per cents, respectively. Since at u_{1w} = 0.05, the proposed solution coincides with the solution of Mishra et al. (1985), therefore, it is concluded that the proposed solution is not sensitive to Ed_i, and Ed_i does not provide a cut off value about the accuracy of the solution.

The magnitude of $t \frac{\partial}{\partial t}[A_1(t)/A_s]$ has decreased continuously with decrease in u_{1w}. A perusal of figures 2,3,4 and 5 reveals that for $t \frac{\partial}{\partial t}[A_1(t)/A_s]$ less than 1.0, the proposed solution does not deviate appreciably from the exact one. In fact, at u_{1w} = 0.05 for T_1/T_2 =0.1, 1.0 and 10.0 the magnitude of the parameter was computed to be 0.6663, 0.1569, and 0.622, respectively.

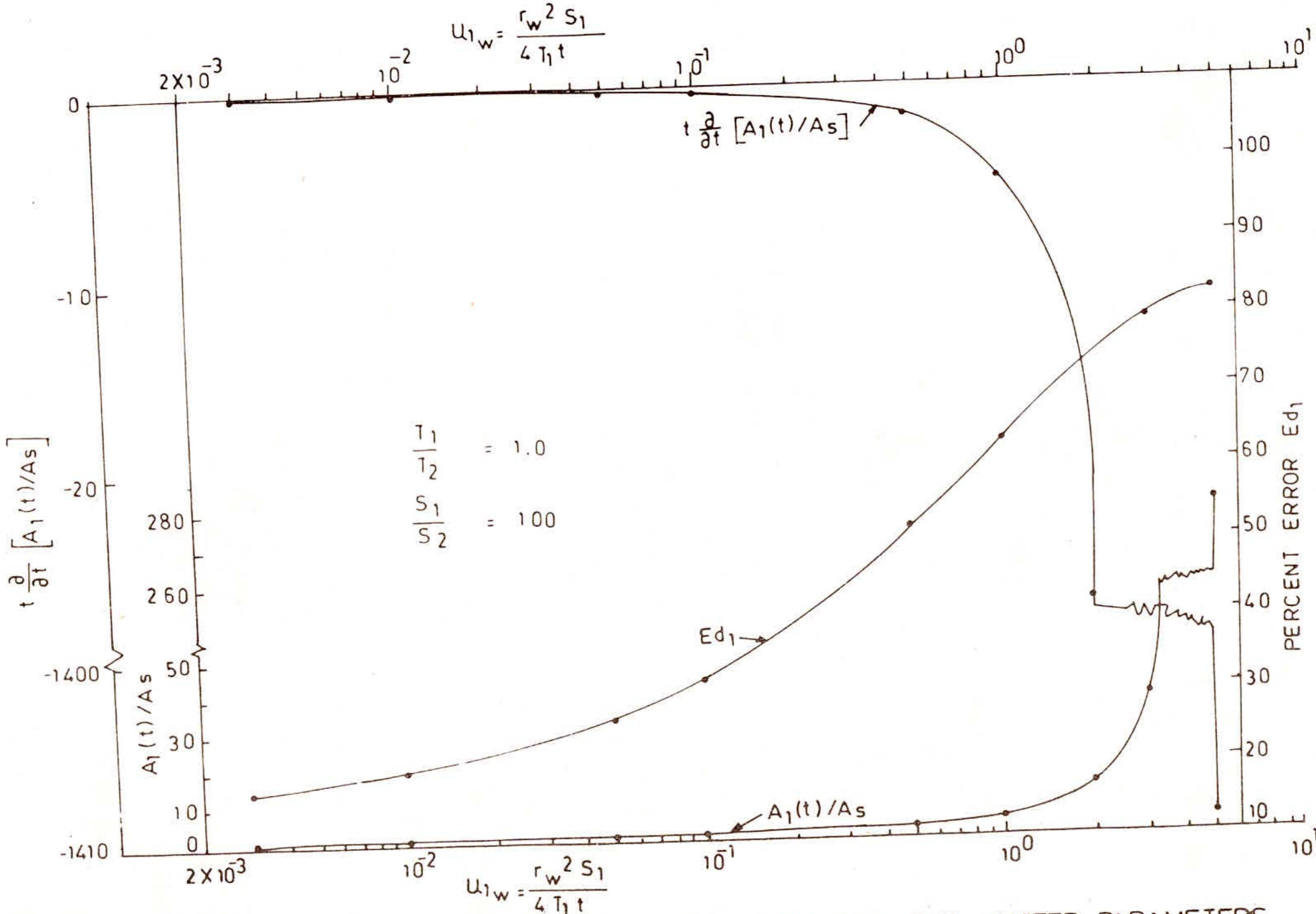

FIG. 4 - VARIATION IN DIFFERENT TERMS OF THE SOLUTION WITH TIME FOR AQUIFER PARAMETERS $T_1/T_2 = 1{\cdot}0$, $S_1/S_2 = 100$

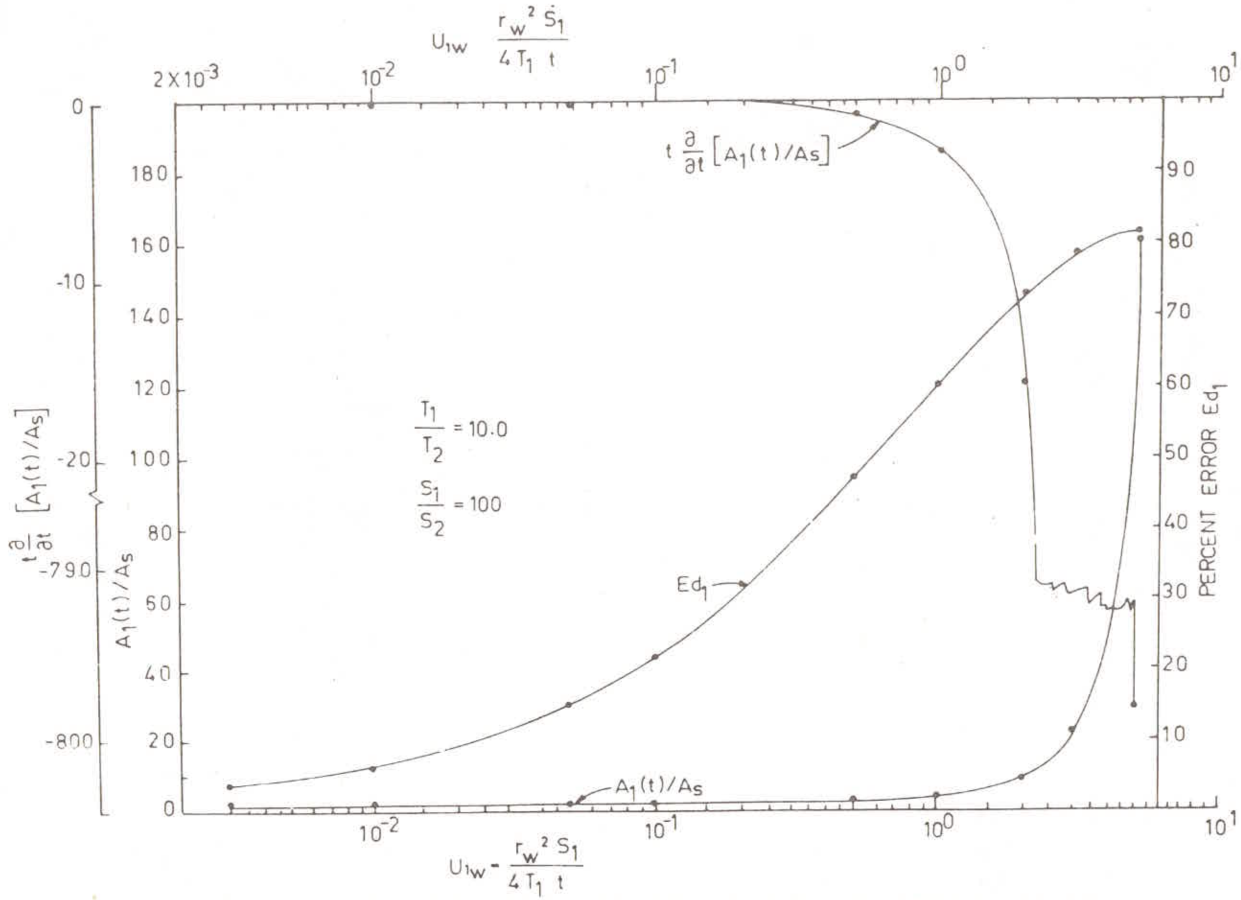

FIG. 5 – VARIATION IN DIFFERENT TERMS OF THE SOLUTION WITH TIME FOR AQUIFER PARAMETERS $T_1 / T_2 = 10$ $S_1 / S_2 = 100$

The function A_s given by (27) is the steady state value of $A_i(t)$. The curves for $A_1(t)/A_s$ in figures 3,4, and 5 real that the function $A_1(t)$ approaches to A_s faster when the aquifer diffusivities D_1 and D_2 are closer to each other.

Computations of $Q_1(t)/Q$ employing equation (17) for two aquifers were also made for $S_1/S_2 = 20$ and $T_1/T_2 = 0.1$, 1.0, and 10.0 Fig.6 shows a plot of $Q_1(t)/Q$ against $\log[(T_1+T_2)t/((S_1+S_2)r_w^2)]$ from the present analysis as well as those of Mishra et al. (1985) and Khader and Veerankutty (1975). The value of u_{1w} corresponding to $\log[(T_1+T_2)t/(S_1+S_2)r_w^2]$ for the three curves are also shown on separate abscissae in the figure. It is evident from the figure that the results of the present analysis match well with those of Mishra et al. (1985) for all values of $u_{1w} < 0.02$. At $\log[(T_1+T_2)t/((S_1+S_2)r_w^2)]=1.1171$ for $T_1/T_2 = 0.1$ the present result is different from that of Mishra et al. (1985) because it corresponds to $u_{1w} = 0.2$.

The curves of the present and Mishra et al. analyses deviate from those of Khader and Veerankutty (1975) at short times i.e. for $\log[(T_1+T_2)t/((S_1+S_2)r_w^2)] < 5$ which corresponds to u_{1w} of the order of 10^{-6}. This difference might result due to limitations associated with the Schapery approximation used by Khader and Veerankutty (1975) to invert the Laplace transform. Since the solutions of Khader and Veerankutty and the present author are valid for $u_{1w} < 0.000001$ and $u_{1w} < 0.05$ respectively, therefore, the proposed solution is superior to the former one.

It may be noted here that in all the six cases of dimensionless computations shown in figures 2 and 6, $u_{1w} > u_{2w}$. Thus it is concluded that equations (3) are valid analytic solution to the system of parabolic partial differential equations (1) so long all $u_{1w} < 0.05$. Consequently, (24) could be substituted for exponential integrals in (3), (9), (10), and (17) while doing computations.

EXAMPLE COMPUTATION

Using the proposed analytic solution it is easy to compute drawdown in each aquifer with the help of the tabulated values of $W(u_i)$(Hantush 1964, p. 322) and a pocket calculator. Such a computation has been reported in Table 1 (example of Mishra et al. 1985) for a well drawing water at a rate of 1000 m^3/day from two aquifers. The other data were $r_w = 0.1$m, $S_1/S_2=100$, $S_1=0.01$, $T_1/T_2=0.5$, $T_1=350$ m^2/day. The table also shows the computed drawdown by Mishra et al. as $s_{im}(r,t)$. It may be

Table 1. Comparison of drawdowns computed from the proposed solution and the solution of Mishra et al (1985)

Time in days	Drawdown in metres: At the well face $s_{1m}(r_w,t)$	$s_1(r_w,t)$	Percent Error	At r = 10m in the first aquifer $s_{1m}(10,t)$	$s_1(10,t)$	Percent Error	At r = 10m in the second aquifer $s_{2m}(10,t)$	$s_2(10,t)$	Percent Error	E_{d1} in per cent	E_{d2} in per cent
1	1.4442	1.4443	-6.92×10^{-3}	0.6072	0.6065	0.12	0.8158	0.8161	-0.04	15.01	13.35
6	1.5825	1.5826	-6.32×10^{-3}	0.7582	0.7580	0.03	0.9477	0.9479	-0.02	13.94	11.81
11	1.6292	1.6292	0	0.8086	0.8081	0.06	0.9925	0.9926	-0.01	13.71	11.35
16	1.6580	1.6581	-6.03×10^{-3}	0.8396	0.8393	0.04	1.0202	1.0204	-0.02	13.47	11.10
21	1.6789	1.6790	-5.96×10^{-3}	0.8621	0.8615	0.07	1.0403	1.0405	-0.02	13.37	10.93
26	1.6954	1.6954	0	0.8798	0.8794	0.05	1.0561	1.0563	-0.02	13.23	10.79
31	1.7089	1.7089	0	0.8943	0.8939	0.04	1.0692	1.0694	-0.02	13.14	10.61
36	1.7204	1.7204	0	0.9066	0.9062	0.04	1.0802	1.0804	-0.02	13.08	10.59
41	1.7304	1.7304	0	0.9173	0.9169	0.04	1.0899	1.0901	-0.02	13.04	10.51
46	1.7392	1.7392	0	0.9267	0.9264	0.03	1.0984	1.0986	-0.02	12.96	10.44
51	1.7471	1.7472	-5.72×10^{-3}	0.9352	0.9349	0.03	1.1130*	1.1063	00.60	12.93	10.36

* Seeing the trend value looks to be misreported

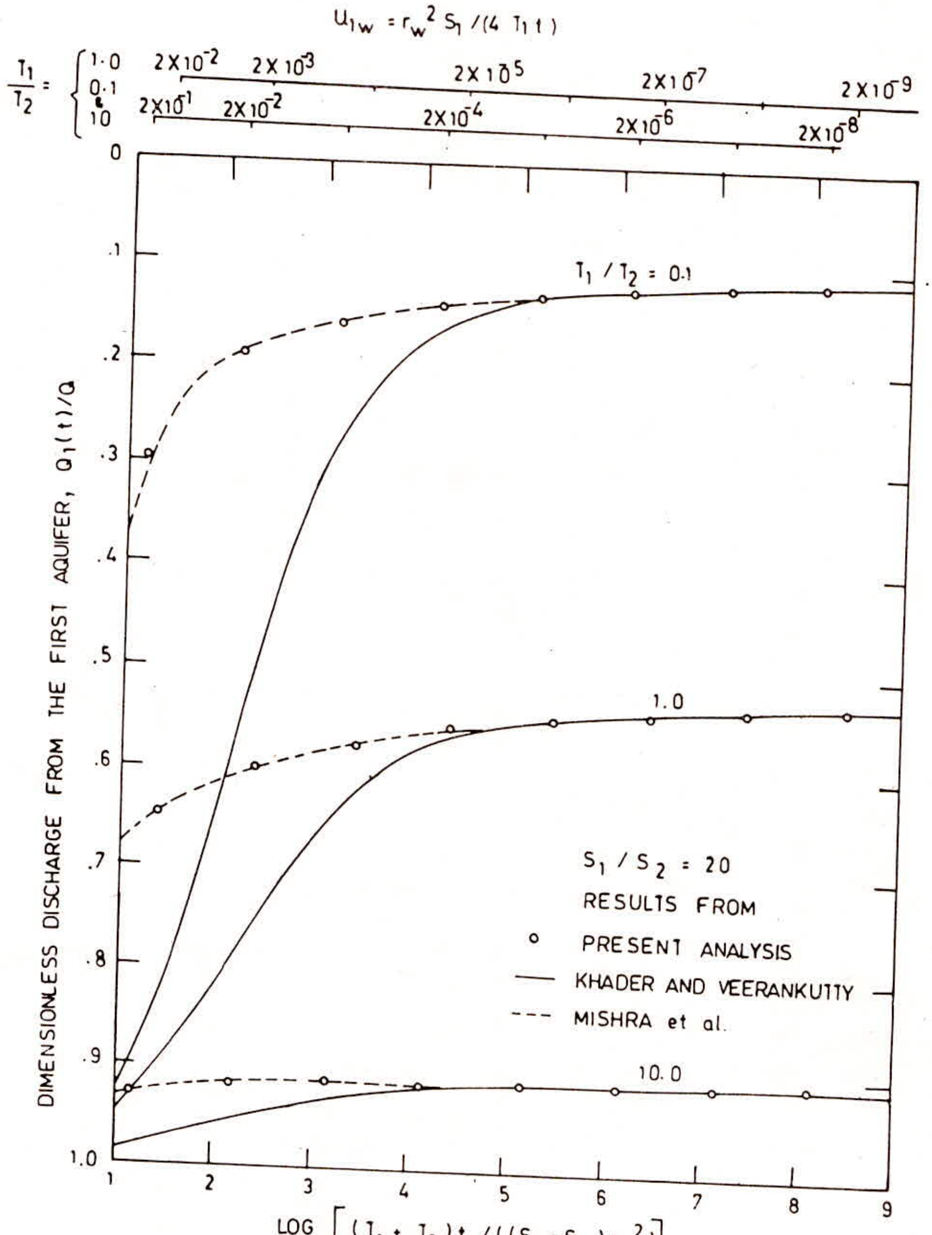

FIG. 6 - COMPARISON OF RESULTS FROM THE PRESENT ANALYSIS WITH THOSE OF MISHRA et al (1985) AND KHADER AND VEERANKUTTY (1975)

remembered that the drawdown at the well face in both the aquifers is the same. In addition to the drawdown the magnitude of the error $E_{di}=[-\ t \frac{\partial}{\partial t} A_i(t)\ W(u_{iw})]/[A_i(t)e^{-u_{iw}}]$ were also computed. The maximum values of E_{d1} and E_{d2} were found to be 15.01 and 13.55 per cents, respectively at 1 day. At this time the dimensionless parameters were $u_{1w}=7.1428 \times 10^{-8}$ and $u_{2w}=3.5714 \times 10^{-10}$. A perusal of the table indicates that even for such high value of E_{d1} and E_{d2}, the error in drawdown computed from the proposed solution is negligible.

SUMMARY AND CONCLUSION

Analytic solution for unsteady ground water flow to a fully penetrating well installed in n-confined aquifers (n is an integer) separated by aquicludes has been developed. The solution is approximate as it fails to satisfy the differential equation at short time. However for $r_w^2 S_i/4T_i t < 0.05$ it is found to be valid as the results of the proposed solution are in close agreement with those of Mishra et al. (1985). When the diffusivities of the aquifers are equal the solution reduces to Theis equation. The analysis reveals that at any time the drawdown difference between two radial distances in an aquifer follows a Thiem type equation in which the transmissivity of the Thiem's equation is replaced by an effective transient transmissivity of the aquifer. However, when the flow reaches steady state the Thiem like equation reduces to Thiem's equation.

An explicit relationship for the ratio of discharge from the ith aquifer to the total discharge of the well has been developed. It reveals that (i) the contribution of an aquifer to the well discharge during transient state is a function of the aquifer diffusivity, (ii) at the start of pumping almost total well discharge is drawn from the aquifer having the least aquifer diffusivity , and (iii) at steady state the ratio is equal to the composite transmissivities of the aquifers. A consequence of (iii) is that the ratio of the discharges from the aquifers is the same as the ratio of their transmissivities. The latter result also holds good at all times, for a well installed in aquifers of equal diffusivities.

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