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OPTIMIZATION AND PROGRAMMING TECHNIQUES FOR RESERVOIR OPERATION

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CONTENTS

LIST OF FIGURES

ABSTRACT

Optimization is one of the most powerful and popular technique for solving various problems associated with the operation of a reservoir. During the past few years, its use has grown tremendously due to wider availability of computer and the solution techniques.

In the present report a comprehensive review of optimization techniques as applied to study the various aspects of operation of reservoirs is made. The theories of linear programming, non linear programming and dynamic programming have been discussed in detail. This follows a review of important works which have used linear programming and dynamic programming as solution techniques for analysis and solution of problems of reservoir operation. Multi objective optimization is also becoming popular nowadays. This aspect has also been discussed and two important techniques for solving multi objective optimization problems have been discussed. A comprehensive bibliography is given at the end for reference purposes.

1.0 RESERVOIR OPERATION

1.1 Introduction

Dams are constructed across the streams to give rise to storage space. This storage space is used to regulate the natural flow of the stream. This regulation may change the temporal and spatial availability of water. The water stored in a reservoir may be diverted to far away places by means of pipes or canals resulting in spatial changes or it may be stored in the reservoir and released later for beneficial uses giving rise to temporal changes.

A reservoir which is operated to serve only one purpose is called a single purpose reservoir. This purpose may be flood control which requires that the reservoir must be operated so as to protect a downstream damage centre by storing the water during high flow periods and releasing it later. In this case the reservoir is called a flood control reservoir. After the passage of the flood, the reservoir is emptied as soon as possible to prepare for next flood event. The operation problem is the regulation of spillway gates so that the downstream flow is less than safe discharge(the discharge at which there is very little or no damage) to the extent possible. For big reservoirs, the problem is slightly simple because small to medium floods can be easily absorbed but for smaller ones it is quite tedious as judicious operation of gates by determining how much to release and when to release has very much influence on the flood moderation.

A single purpose reservoir may be operated to cater for irrigation, water supply, hydroelectric power, navigation or recreation etc. These purposes have one thing in common; it is required to have as much water as possible in the reservoir. This is in clear contrast to flood control operation. These purposes are called conservation purposes. In case of irrigation operation, the reservoir is required to supply water for crop irrigation to augment the water available from rainfall and ground water. The demand for water will vary depending upon these factors as well as the type and extent of cropping in the command area. The crop water requirement also varies depending upon the stage of crop growth. As the water use benefit function is highly non-linear, it is very important to optimally determine as to when water should be supplied, particularly in case of scarcity.

The hydroelectric energy generation from a power plant of a reservoir depends upon two factors- the head available and the discharge passing through turbines. Thus, if the reservoir level is high, less water will be required to generate a particular amount of energy while comparatively more water will be required at low heads. Here the operating decision is how much energy should be generated to best meet the targets.

For a reservoir which serves for recreation, benefits are obtained from the use of lake for boating, picnic, swimming and other water sports etc. These demands require a high water level in the reservoir and sufficient surface area of the lake. Further, the variations in the level of reservoir must be minimum. Large variations in the lake levels cause bank sloughing and interruption in picnic facilities because these have to be frequently shifted.

The operation of a multipurpose reservoir is aimed towards meeting more than one of the above purposes. The complexity of the problem depends upon the nature of the purposes which are combined together. For example, let us consider the case of a reservoir serving for irrigation and municipal and industrial water supply. Both these purposes require the reservoir to be as much full as possible. The decision to be taken is as to how much water should be released from the storage to meet the demand and in what proportion it should be divided between two purposes. Similarly, in case hydroelectric power and irrigation needs are combined, the water released from the reservoir can be passed through turbines first and then used for irrigation. But many times two or more conflicting purposes are combined as in case of a reservoir serving for irrigation and flood control. It is a well known fact that in India upto 80% of annual flows occur during four monsoon months . Hence this is the period during which the operator has to fill the reservoir so that he can meet the demands during the rest of the year. However, this is also the time when major floods occur and in case a major flood occurs when the reservoir is already full, it may not be possible to moderate it and the safety of the structure itself may be under risk. On the other hand the reservoir can not be kept empty with the aim to moderate the likely floods. These floods may never come and the reservoir may remain empty. Thus the bottom line of the argument is that whatever is safe from one point of view is equally unsafe from other points of view when the purposes are conflicting.

1.2 The Need for Optimization

It is clear from the above discussion that the problem of reservoir operation arises mainly because of scarcity of water and conflicts among purposes. If plenty of water is available all the time, there is virtually no serious management problem. Secondly, it is also seen here that at each time, there are a large number of possible alternative decision options available. However, it is not possible to evaluate each of them without a computing aid. Even with it, unless calculations are performed in a systematic manner, the solution may be impossible to obtain. For example, if the attempt is made to get the solution by evaluating each discrete point in the feasible space and then comparing the outcome, computer memory requirements will exceed what is available right now even on major computer systems. Further, the solution, if possible,will be too expensive to obtain.

In the optimization techniques, an initial solution is chosen first which is feasible. Thereafter, a systematic search is made to obtain a better solution until no further improvement is possible. The algorithms are designed in such a way that a definite answer is obtained in a finite number of steps.

1.3 Scope of the Present Study

The present report deals with the problems associated with operation of reservoirs and the status of optimization models developed for their solution. In the beginning, a detailed description of a few optimization techniques which are commonly used in the field of reservoir operations has been given. After this, a comprehensive review of models proposed by various investigators to solve the problems encountered regarding operation of reservoirs is made. A bibliography of the related works is given at the end.

2.0 OPTIMIZATION TECHNIQUES

2.1 Introduction

Optimization is the science of choosing the best from amongst a number of possible alternatives. In many engineering problems, a situation arises in which there are many ways of doing a particular thing. For example, a number of alternative designs may be available to serve the required need, a number of management decisions may be available to increase the production and a number of release decisions may be available to cater for irrigation and hydro-electric power. Naturally, the result attained in each case will be different and hence it is required to evaluate each alternative and then choose the best from the point of view of interest, say economical or physical or convenience etc.

The complexity of these type of problems goes on increasing with the number of factors affecting a particular choice. For simple problems, the analysis can be performed manually but the computations become unmanageable manually when the number of factors become large and the optimum choice in such cases is to use a digital computer.

Mathematically, an optimization problem can be stated as:

$$\text{Find } X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad \ldots (1)$$

which minimizes f (X)

Subject to the constraints

$$g_j (X) \geq b_j, \; j = 1,2.......,m \qquad \text{... (2)}$$

$$\text{and } l_j(X) = b_j \; j = m+1, m+2........,p \qquad \text{... (3)}$$

where,

X = a n-dimensional vector called decision vector,

f (X) = objective function,

$g_j(X)$ = inequality constraints, and

$l_j(X)$ = equality constraints.

The decision vector represents the variables to be manipulated to obtain the solution. As an example, release from the storage is a decision variable of a reservoir operation problem. The decision variables are chosen such that the operator's objective, f(X), is optimized, i.e.,either minimized or maximized. Generally, the objective function either represents benefits in which case it is maximized or it represents costs which are to be minimized. It can be easily seen that maximization of a function is analogous to minimization of its negative. Thus, in a reservoir operation problem, the aim may be to maximize the benefits by deciding the amount of water to be released. But in many cases it may not be possible to release a particular amount of water because of limited capacity of outlet structures or because that much water may really not be available. In other words, the decision variable is forced to take value within a specified range.If this is the case then the problem is said to be a constrained one. It may happen that constraints may force the decision variable to have value less than a upper limit (say release restriction because of limited capacity of outlet works), or they may force it to have a value greater than a lower limit(say a binding that release must always be greater than a minimum value), or both.

These types of constraints are known as inequality constraints since the left hand side and right hand sides of the constraints need not be equal. However, it may so happen that the condition of equality has to be satisfied. Continuity equation, for example, appears as equality constraint in many problems. It specifies that the initial storage plus inflow minus releases and losses must be equal to the end of period storage.

An optimization problem, in which no constraints are present, is called an unconstrained optimization problem. This condition rarely arises in practice where constraints are almost always present and the problem is of constrained optimization.

A solution of the optimization problem which satisfies all constraints is called a feasible solution. A feasible solution, however, may or may not be optimum. If no further improvement to a feasible solution is possible, the solution is called optimum solution. If it is not possible to get any feasible solution, the problem is termed infeasible. Further, it may happen that more than one combination of decision variables may give same value of objective function which may also be the best value, in such cases all the solutions are called alternative optimum. Sometimes due to wrong formulation it so happens that the objective function can be increased(or decreased) as much as one wishes. The corrective step is to check the formulation, particularly constraints.

The optimum seeking methods are also known as mathematical programming techniques. Depending upon the nature of the problem, the available optimization techniques can be classified in several ways, viz.,linear or nonlinear optimization, deterministic or stochastic optimization, constrained or unconstrained optimization etc. A useful way of classifying the techniques is shown in figure 1. For the purpose of this report, the discussion of methods is being made under following three major heads:

(a) Linear Programming,

(b) Nonlinear Programming, and

(c) Dynamic Programming.

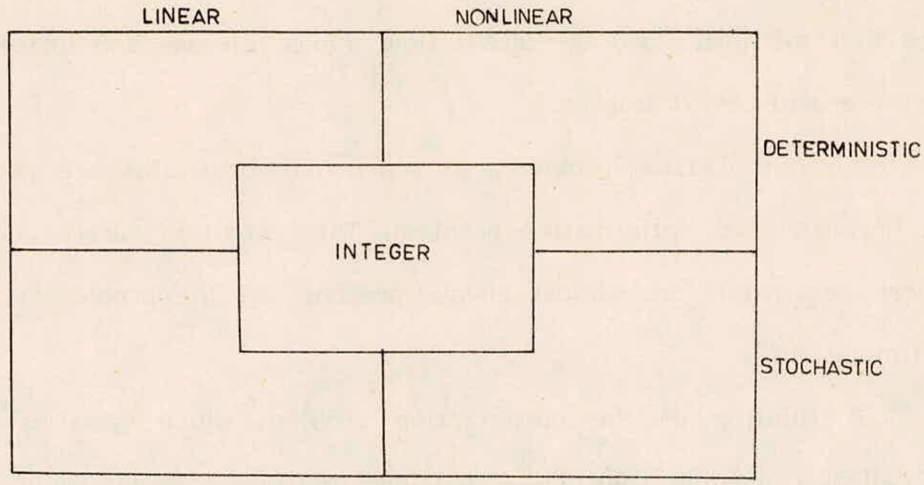

FIGURE- 1 CLASSIFICATION OF OPTIMIZATION TECHNIQUES

2.2 Linear Programming (LP)

The optimization problems in which the objective function and constraints are linear in nature alongwith the condition that the decision variables are positive are termed as Linear Programming problems.

The standard and expanded form of a LP problem is:

$$\text{Min } Z = \Sigma\, c_i\, x_i \qquad \ldots(4)$$

subject to

$$a_{11}\, x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n \geq b_1$$

$$a_{21}\, x_1 + a_{22}\, x_2 + \ldots\ldots + a_{2n} x_n \geq b_2$$

$$a_{m1}\, x_1 + a_{m2}\, x_2 + \ldots\ldots + a_{mn}\, x_n \geq b_m \qquad \ldots(5)$$

$$\text{and, } x_i \geq 0 \quad i = 1, 2, \ldots\ldots, n.$$

Here x_i are decision variables, c_i are cost coefficients (or benefit coeff.)

of the particular problem, a_{ij} and b_i are coefficients. The coefficients c_i represent cost incurred by increasing the x_i decision variable by one unit. For example, it may represent additional economic loss by increasing the flood release by one unit. The right hand side of constraint equations represents resource availability.These arise due to limited availability of a particular resource, say water. The a_{ij} coefficients are called technological coefficients and quantify the amount of particular resource i required per unit of j^{th} activity. Further, a constraint of ($\geq$) type can be easily converted to a ($\leq$) type by multiplying (-1) throughout the equation.

An inequality constraint of ($\geq$) can be converted to equality type by introducing a variable $s_i \geq 0$.

Thus the constraint

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \geq b_1 \qquad \dots (6)$$

is equivalent to

$$a_{11} x_1 + a_{12} x_2 + \dots a_{1n} x_n - s_1 = b_1 \qquad \dots (7)$$

The variable s_1 is called as surplus variable.

Similarly, an inequality constraint of the type ($\leq$) can be converted to equality type by introducing a slack variable s_1. Hence the constraint

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1 \qquad \dots (8)$$

can be written as

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n + s_1 = b_1 \qquad \dots (9)$$

2.2.1 Matrix representation of a LP problem

In matrix notations, a LP problem may be presented as:

$$\text{Min } Z = C^T X$$

subject to

$$A X \geq b \qquad \dots (10)$$
$$X \geq 0$$

where,

$$X = [\, x_1\ x_2 \cdots\cdots x_n]$$

$$C^T = [\, c_1\ c_2 \cdots\cdots c_n\ ]^T$$

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{vmatrix}$$

2.2.2 Standard form of a LP problem

A LP problem is written in standard format as

$$\text{Min } Z = c_i\ x_i \qquad \ldots(11)$$

subject to

$$a_{11}\ x_1 + a_{12}\ x_2 + \cdots\cdots + a_{1n} x_n = b_1$$

$$a_{21}\ x_1 + a_{22}\ x_2 + \cdots\cdots + a_{2n}\ x_n = b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2}\ x_2 + \cdots\cdots + a_{mn}\ x_n = b_m$$

and $x_1\ ,\ x_2, \cdots\cdots\ x_n \geq 0$

If the number of variables n is equal to the number of constraint equations m, the problem has a unique solution, if it exists. If m >n, and if m-n equations are not redundant, it has a solution only in the least square sense.

If m < n, then we can set (n-m) variables equal to zero and solve the m equations for m variables. However, there will be ${}^nC_m$ such solutions. Each of these solutions is called a basic solution. The (n - m) variables which have been set equal to zero are called nonbasic variables; remaining n variables are called basic variables.

A basic solution which satisfies all the constraints is called a basic feasible solution and any such solution which provides minimum (or maximum) value of the objective function is called an optimum solution. The feasible region and constraints are shown in figure 2.

Suppose in a particular problem, n = 20 and m = 10, then the number of possible basic solutions will be $^{20}C_{10}$

$$\text{or} \qquad \frac{20!}{(20-10)!\ 10!} = 184756$$

Hence to solve this problem, 184756 solutions will be required to be obtained and compared. This is formidable task even with the help of a fast digital computer. A very efficient method was developed by Dantzig which is called Simplex Method. Before discussing the Simplex Method, graphical solution of a LP problem is being discussed.

2.2.3 Graphical solution of a LP problem

Here a LP problem in two dimensions will be discussed.

$$\text{Max } Z = 2x_1 + x_2$$

subject to

$$2x_1 - x_2 \leq 8 \qquad \ldots(13)$$

$$x_1 + x_2 \leq 10$$

$$x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

In the figure 3, constraints are plotted against the coordinate axes x_1 and x_2. The non-negativity constraints are plotted as the axes themselves. To mark the constraint $2x_1 - x_2 \leq 8$, we plot a straight line $2x_1 - x_2 = 8$. Similarly, we plot lines $x_1 + x_2 = 10$, and $x_2 = 7$ to mark second and third constraints. The feasible region can be easily delineated and is shown by hatched lines.

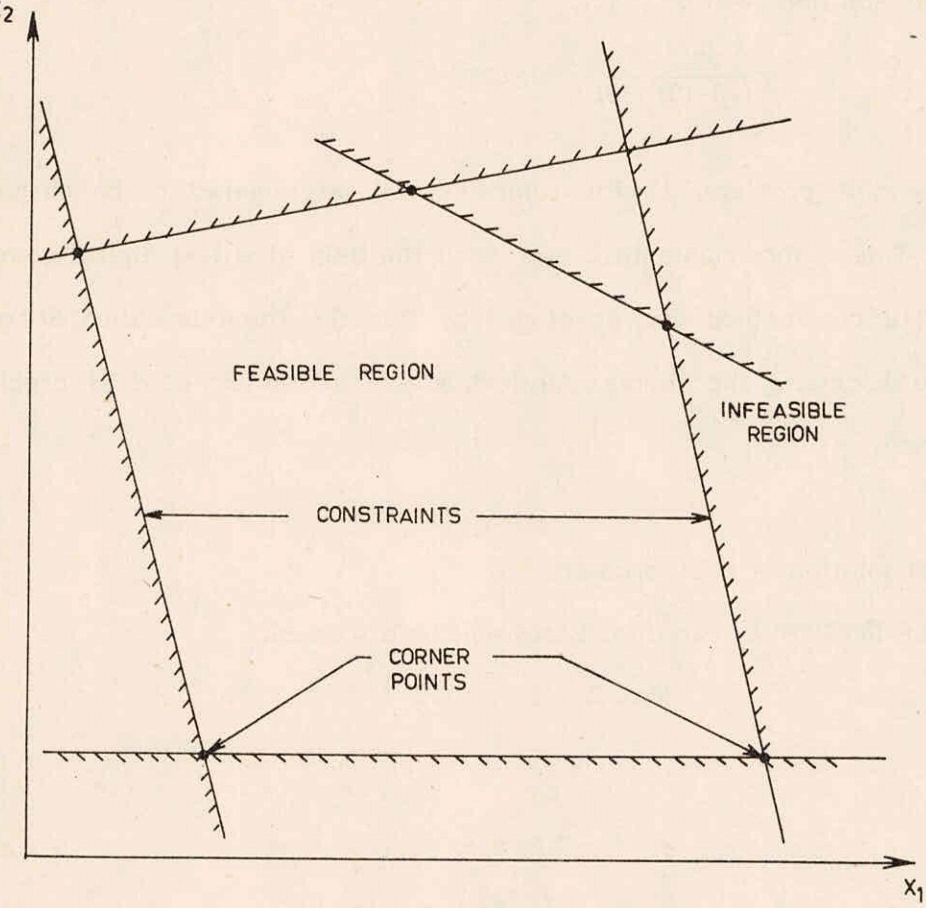

FIGURE - 2 FEASIBLE REGION AND CONSTRAINTS

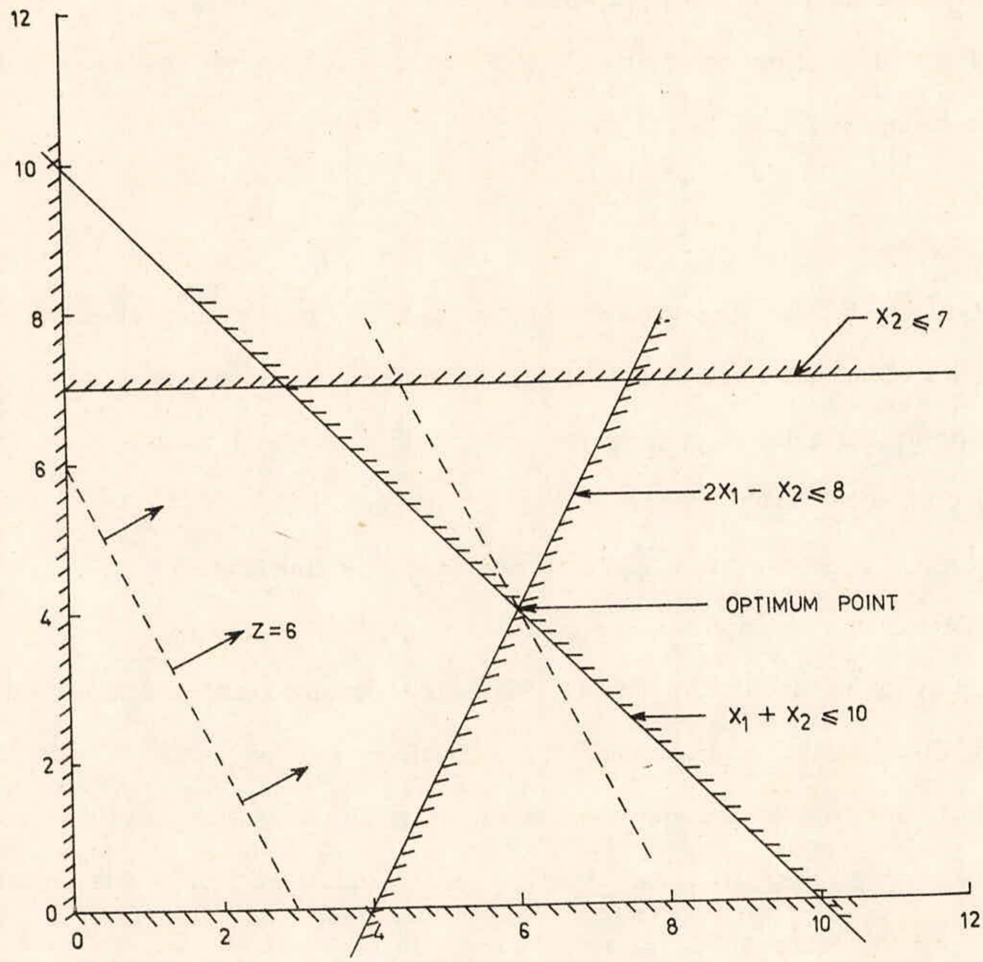

FIGURE - 3 GRAPHICAL SOLUTION OF A LP PROBLEM

Now we start with a particular value of objective function, say 6 and plot the line $2x_1 + x_2 = 6$. Since it is a maximization problem, the objective function line is shifted forward as far as possible while ensuring that at least one point lies in the feasible region. It can be seen that the farthest point upto which we can go is a point (6, 4). Hence, this is the optimum point at which the objective function is equal to 16 and $x_1 = 6$ and $x_2 = 4$.

A closer inspection of figure 3 will show that the optimum point will always be a corner point.

2.2.4 Simplex method

To begin with, we first transform the problem into canonical form. The characteristics of canonical form are:

(a) The basic variables have positive unit coefficients and only one of them (different each time) appears in each equation.

(b) The basic variables do not appear in the objective function.

(c) The RHS of constraints must be positive.

One way is to arbitrarily choose the basic variables and use a technique like Gauss Elimination to transform the equations in the canonical form. If the equations contain only the slack variables, these can be automatically considered as basic variables. But in case problem has surplus variables and equality constraints, we introduce artificial variables in the equation. An auxilliary objective function is now formed which is equal to the sum of artificial variables. The computations are demonstrated using following example.

$$\text{Min } Z = x_1 + x_2$$

subject to

$$x_1 + 2x_2 \geq 5 \qquad \ldots (14)$$

$$2x_1 + x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

Writing this problem in standard form by introducing surplus variables

$$\text{Min } Z = x_1 + x_2$$

subject to

$$x_1 + 2x_2 - x_3 = 5 \tag{15}$$

$$2x_1 + x_2 - x_4 = 4$$

$$x_1, x_2, x_3, x_4 \geqslant 0$$

Now in this problem, the number of variables n is 4 and number of constraints m = 2. Hence, there will be two basic variables which could be chosen arbitrarily. If the coefficients of x_1 and x_2 were + 1, the equations would have been in the canonical form. This not being the case, we introduce two artificial variables. The idea is to avoid laborious computations using Gauss Elimination method. The problem now looks like

$$\text{Min } Z = x_1 + x_2$$

subject to

$$x_1 + 2x_2 - x_3 + x_5 = 5$$

$$2x_1 + x_2 - x_4 + x_6 = 4 \tag{16}$$

$$x_1, x_2, \ldots x_6 \geqslant 0$$

The auxiliary objective function is

$$\text{Min } W = x_5 + x_6$$

$$= (5 - x_1 - 2x_2 + x_3) + (4 - 2x_1 - x_2 + x_4)$$

$$= 9 - 3x_1 - 3x_2 + x_3 + x_4$$

Now the problem in canonical form is

$$\text{Min } Z = x_1 + x_2$$

$$\text{Min } W - 9 = -3x_1 - 3x_2 + x_3 + x_4 \tag{17}$$

subject to

$$x_1 + 2x_2 - x_3 + x_5 = 5$$

$$2x_1 + x_2 - x_4 + x_6 = 4$$

$$x_1, x_2 \ldots, x_6 \geqslant 0$$

This method is called two phase simplex as we have two objective functions. The first phase aims at minimization of the auxiliary objective function. If as a result of this phase, this function can not be made zero then the problem is infeasible and the algorithm is terminated. If W = 0, then the optimization of the main function is taken up. For ease of computation, the Simplex Tableau is formed as follows:

RHS	x_1	x_2	x_3	x_4	x_5	x_6
5	1	2	-1	0	1	0
4	2	1	0	-1	0	1
-Z=-0	1	1	0	0	0	0
-W=-9	-3	-3	1	1	0	0

2.2.5 Computational steps of Simplex Method

(i) After simplex tableau is formed, it is checked whether the objective function will improve by replacing a basic variable. A solution will be optimal if all the cost coefficients are positive or zero in a minimization problem or are negative or zero in a maximization problem.

Thus ,if optimality is not satisfied then the variable which will improve the objective function at the fastest rate, i.e.,for which cost coefficient is most negative(for minimization) or most positive(for maximization) is brought in the basis. The decision is arbitrary in case of tie. Let this variable be x_r.

(ii) Now for this variable, we take b_i / a_{ir} ratio for each constraint row i and the minimum ratio determines the row in which this variable will have unit coefficient. Corresponding variable from this row(which was a basic variable) will leave basis. The equations are again converted into canonical form by suitable row and column operations.

These steps are repeated until an optimal solution is found.

2.2.6 Concept of dual

Associated with every linear programming problem called primal, there is another problem called its dual.

Let the primal be

$$\text{Min } Z = C^T X \qquad (18)$$

subject to

$$A X \geqslant b$$

$$X \geqslant 0$$

Then the corresponding dual will be

$$\text{Max } Z_1 = b^T Y \qquad (19)$$

Subject to

$$A^T Y \leqslant C$$

$$Y \geqslant 0$$

Some important relations between primal and dual are:

(1) If the primal is a maximization problem, the dual will be a minimization problem and vice versa.

(2) The dual of a dual is primal.

(3) If a finite solution exists for the primal, same is also true for dual.

(4) For each variable in primal, there exists a constraint in dual and vice versa.

(5) If the primal has an unbounded solution then the dual will either have an unbounded solution or will be infeasible.

(6) Let the primal be

$$\text{Min} = 5 x_1 + 4 x_2 + x_3$$

Subject to

$$- 2x_1 + 3x_2 + x_3 \geqslant 10$$

$$2x_1 - x_2 + 3x_3 \geqslant 15 \qquad (20)$$

$$x_1 , x_2, x_3 \geqslant 0$$

The the dual will be

$$\text{Max } Z_1 = 10\,y_1 + 15y_2$$

Subject to

$$-2y_1 + 2y_2 \leq 5$$

$$3y_1 - y_2 \leq 4$$

$$y_1 + 3y_2 \leq 1 \qquad (21)$$

$$y_1, y_2 \geq 0$$

Similar to primal simplex, a technique called dual simplex can be used to solve a dual problem. If a finite solution exists, then the optimal solution for primal and dual are same.

In dual problems, y is a vector of variables called shadow prices or opportunity costs. These define the contribution of each constraint to the objective function. The shadow prices reflect as to how much more output can be obtained by releasing a constraint by one unit of any input resource. Further, if the optimal solution does not completely use any of the resource then the shadow price of that resource must be zero and any excess quantity of this resource has no economic value since whatever is already available has not been fully utilized. If the price of a resource is less than its shadow price in a given situation, it is desirable to buy more of that resource and expand the production. The significance of duality and shadow price is that it gives added insight into the problem.

Further, many times solution of the dual may be computationally more efficient than that of the primal, say in case where primal contains a large number of constraints.

2.2.7 Transportation Problem

Transportation problems are special type of linear programming problems. The aim is to seek the optimum transportation plan of a commodity from a number of source nodes to a number of destination nodes. This problem, most commonly arises when a big company has several production centers, may be all over the world, and a number of demand centers or warehouses. In simple transportation models, there is only one commodity. More than one commodities can be considered in a complex transportation model.

Considering the case of a single commodity, the demand at a particular node can be met with from a number of sources. The required input includes the production at each supply center, demand at each destination point and the unit cost of transportation from each source to each destination. If the total supply is equal to the total demand, the problem is said to be a balanced one. The solution consists of the amount of commodity to be transported from each source node to each destination node such that the total cost of transportation is minimized.

Being a linear programming problem, the transportation problem can be easily solved using Simplex Method. However, because of special nature of the problem, solution techniques which are far more efficient than Simplex are available. As usual, first a basic feasible solution is obtained which is subsequently improved.

Three procedures are available to get a starting feasible solution for a transportation problem. These are North-West Corner rule, Least Cost method and Vogel's Approximation technique. Out of these three, the Vogel's Approximation technique provides the best starting solution and many times, little or no improvement to it is required to obtain the optimal solution.

The transportation model can be very easily applied to management problem of a water resources system. The reservoir or groundwater pumping nodes are the sources of the commodity water and the agricultural land, cities and industrial areas are the demand nodes. Depending upon the losses during flow and other factors, the unit cost of transportation of water from a supply node to demand node can be determined. However, it may not be possible to supply water from a particular source to a demand node because of physical reasons such as negative slope etc. In such cases very high cost value is assigned to that route. To make the formulation more realistic, upper limits on capacities are imposed for sources and transporting routes. These refer to capacity limits for reservoirs and carrying capacities of channels.

By doing the above exercise, we are infact moving towards very popular algorithms called 'Network Flow Algorithms'. A network consists of a number of nodes and links. A reservoir, a diversion point, and a junction of two rivers are typical examples of the nodes. Further, a node may or may not have a storage capacity. The nodes are interlinked by links. Natural rivers, canals and pipelines form link in a water resources network. In general algorithms, flow in a link can occur in both directions but in case of water resources, generally only unidirectional flow is possible in a link. In such cases, the network is called directed. The continuity equation must be satisfied for each node, i.e.,the flow entering the network added to the initial storage, if any,less the outgoing flow must be equal to the end of the period storage. In the earlier developed techniques, the flow entering in a link was assumed equal to the flow leaving it or in other words, there was no provision of channel losses. This provision was made in more generalized versions later on wherein there is a provision of expressing the channel losses as a percentage of flow entering in the channel. This factor is called gain of the channel and the network with this feature is called network with gains. The technique most commonly used to solve a network problem in water resources is 'Out-of-Kilter 'algorithm. This is nothing but a linear programming formulation where the objective is to minimize the total cost of flow ensuring the satisfaction of demands to the extent possible.

A large number of generalized computer programs are available which use out-of-kilter algorithm. These programs are mostly quite general in the sense that any system configuration can be studied. The inflow to the network and spill from it can only occur at nodes, all the demands are also to be placed at nodes only.

2.2.8 Sensitivity Analysis

The coefficients used in an optimization problem are not sacrosanct numbers and in many engineering applications, it is required to find out how the solution is effected by a change in them. For example, we may have to find that if the cost coefficient of a particular variable changes then whether the optimal solution is changed or not and if it is changed than what is the new solution.The analysis,which is also called post optimality analysis,can be classified in following distinct cases:

(1) Changes in cost coefficients of a)basic variables and b) non-basic variables.

(2) Changes in RHS constants

(3) Changes in technological (a_{ij}) coefficients

(a) of basic vector (b) of non-basic vector

(4) Addition of new variables

(5) Addition of new constraints

Depending upon a particular problem, three types of cases may arise:

(1) The optimal solution may remain unchanged.

(2) It may be possible to obtain the new solution by proceeding from the final Simplex tableau.

(3) The entire problem may have to be resolved to obtain new solution.

The technique of linear programming is very extensively used in solving water resources system problems. One main reason is that very efficient and generalized computer packages are available nowadays. The typical input to these programs consist of specifying the type of the problem, i.e.,whether maximisation or minimisation, and cost coefficients in the objective function. The required information about the constraints include their number, their type (i.e.equality, greater-than-and-equal-to, or less-than-and-equal-to), the coefficients on the left hand side and RHS constants. Output from the program includes message whether optimal solution could be found or the solution is infeasible or unbounded etc.,the value of objective function at optimum point and value of basic variables. Further details such as intermediate computational results etc. can be printed using several options which are usually available.

2.3 Nonlinear Programming

As mentioned earlier, the precondition for application of linear programming is that the objective function and constraints must be linear in nature. However, in many engineering problems this may not be the condition. Further, the possible remedy of linearizing these functions may lead to unwanted distortions in the objective function and constraints and hence inaccurate results. In such cases, the technique of non-linear programming is used to solve the problem.

A nonlinear programming problem can be stated as

$$\text{Min } f(x) \qquad ...(22)$$

$$\text{Subject to} \quad g_i(x) \geqslant 0 \qquad i = 1,2,....p \qquad ...(23)$$

$$h_j(x) = 0 \qquad j = 1,2,... q$$

where x is a vector of variables, g represents all inequality constraints and h represents all equality constraints.

In this section, optimization of functions of single variable will be discussed first followed by unconstrained minimization problems and then constrained minimization problems.

The most fundamental method of solving an optimization problem is based upon differential calculus. The objective function, however, must be continuous and twice differentiable. This technique, though very simple and straightforward, is not very useful and hence is not generally used.

2.4 One Dimensional Search Techniques

In most of the optimization problems, the range of the variable within which the solution lies is known. This information is very useful for one dimensional search techniques which are best applicable to unimodal functions. A function which has only one extreme point in a given interval is called a unimodal function. Two unimodal functions

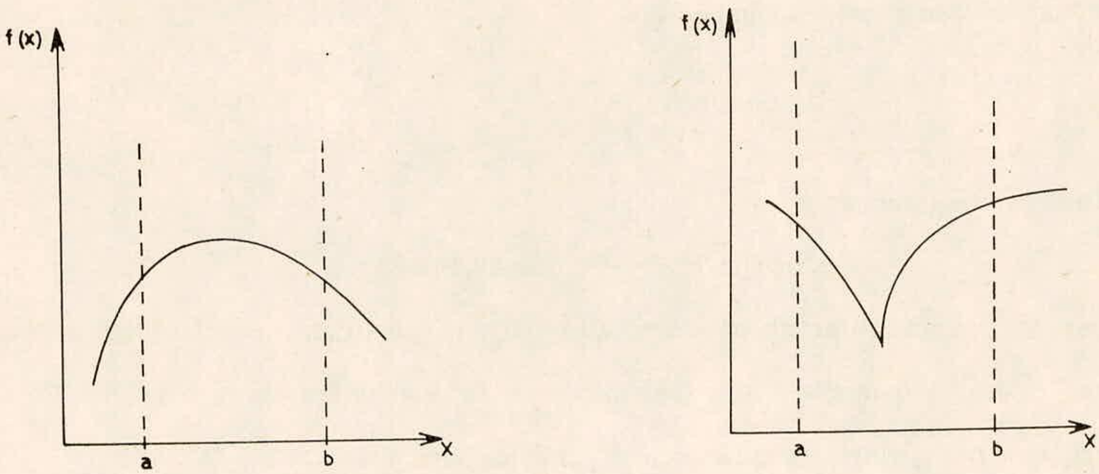

FIGURE - 4 UNIMODAL FUNCTIONS

are shown in figure 4. For a unimodal function, if the function value is given at two points which are on the same side of the optimum than the point which is nearer to the optimum gives better value of the objective function. A unimodal function can be nondifferentiable and/or discontinuous. If it is known that a function is unimodal in a given range, the interval can be reduced to the required degree to bracket the optimum point.

A number of methods are available which can be classified as search methods. These include technique of exhaustive search in which the function is evaluated at a number of points, and sophisticated methods like Fibonacci method and Golden Section method etc. Here only Fibonacci method is being discussed.

2.4.1 Fibonacci method

In this method, the function to be optimized is evaluated at the specified number of points. This method can not locate the exact optimum point but only upto the required accuracy. A sequence of numbers, called Fibonacci numbers is used to select experimental points in this method. These numbers follow a sequence in which the first two numbers are one and thereafter each next number is sum of two previous numbers.

$$F_o = F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2} \qquad n=2,3.... \qquad ...(24)$$

The resulting sequence is

1,1,2,3,5,8,13,21 ,34 ,55,89,144,233......

Let the initial interval of uncertainty U_o in particular problem be between a and b, and the number of experiments to be conducted be n. Now the first two experimental points are placed at x_1 and x_2 such that

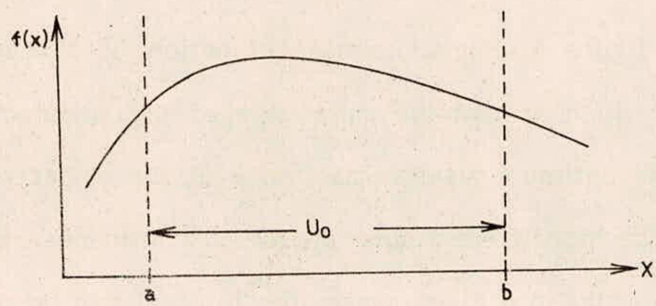

FIGURE - 5 FIBONACCI METHOD

$$x_1 = a + U_1^*$$

$$x_2 = b - U_1^*$$

where

$$U_1^* = \frac{F_{n-2}}{F_n} U_o \qquad \text{...(25)}$$

Let it be a maximization problem. If the function value at x_1 is greater than that at x_2 then the region beyond x_2 is rejected and the new uncertainty interval is from a to x_2 otherwise the region upto x_1 is rejected and the uncertainty interval is from x_1 to b. The new uncertainty region is given by

$$U_1 = U_o - U_1^* = U_o - \frac{F_{n-2}}{F_n} U_o$$

$$= U_o \left(1 - \frac{F_{n-2}}{F_n} \right)$$

or

$$U_1 = U_o \frac{F_{n-1}}{F_n} \qquad \text{...(26)}$$

One experimental point already exists in this interval and the new experimental point is placed at a distance U_2^* from this point where

$$U_2^* = \frac{F_{n-3}}{F_n} U_o \qquad \text{...(27)}$$

Again a part of new uncertainty interval can be rejected using the assumption of unimodality. In this way the uncertainty interval goes on reducing and the procedure is terminated after the required number of iterations is over.

2.5 Unconstrained Optimization

In the one dimensional search methods discussed in the above article, search is made parallel to the coordinate axis. Due to this, the convergence of the algorithm is slow. Sometimes, the method may not even converge or the rate of convergence becomes slow as the optimum point is approached.

To avoid this problem, the search direction is taken as the direction in which the objective function decreases(or increases) most rapidly. The class of methods which use this strategy is known as pattern search methods.

In these methods, usually local exploration near the starting point is made first to learn the behaviour of the objective function in the region around the starting point. This helps in pattern development. Next, moves are made along the most advantageous direction.

Among the available methods, two basic methods, Simplex method and Powell's methods will be discussed in this section.

2.5.1 Simplex method

Although the names are similar, this method is different than the Simplex method for linear programming problems. A Simplex is a geometric figure formed by (n+1) points in n-dimensional space. Thus a triangle in two dimensions is a Simplex and a tetrahedron in three dimensions is Simplex. If the vertex points are equidistant, the simplex is called regular. In this method, the objective function value is compared at the vertices of a simplex and then progress towards optimum is made using three kind of moves-reflection, contraction and expansion. Let us consider a triangle, a simplex in two dimensions as shown in figure 6.

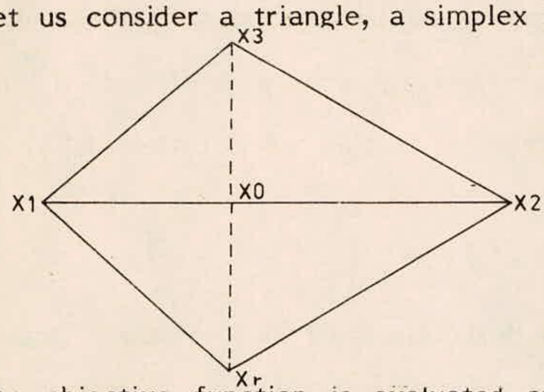

FIGURE - 6 REFLECTION IN A SIMPLEX

The objective function is evaluated at three points x_1, x_2, and x_3. Let this be the minimization problem. As a result of comparison, let x_3 be the point where the function value is maximum. Hence, it is most likely that the point opposite to x_3 will have a smaller value of the objective function. So a new simplex x_1, x_2, x_r is formed by reflecting the point x_3 with respect to the face x_1,x_2.

To get the coordinates of point x_r, first the centroid of all points except the one at which objective function is maximum, is computed.

Defining

$$x_h = \max_{i=1\ldots n+1} f\,(x_i) \qquad \ldots(28)$$

where x_h is the point having maximum objective function value and

$$x_o = \frac{1}{n} \sum_{\substack{i=1 \\ i \neq h}}^{n+1} x_i \qquad \ldots(29)$$

$$x_r = (1+\alpha)\ x_o \quad \alpha\, x_h \qquad \ldots(30)$$

where $\alpha > 0$ is the reflection coefficient. Generally α is taken equal to unity. It may be seen that with the available information, movement from x_3 to x_r will be most favourable.

In some cases, for example when the simplex straddles a valley, the objective function value may be same at x_3 and x_r and we may enter in a loop. In such cases, instead of reflecting the point having highest function value, the point having second highest function value is reflected and the process is continued.

If the objective function value at the reflected point is better, we may expect it to improve further by moving ahead in that direction This step is known as expansion and the coordinates of expanded point are given by

$$x_e = r\, x_r + (1-r)\, x_o \qquad \ldots(31)$$

Where $r > 1$ is called the expansion coefficient. Its usual value is two.

If expansion step is successful, the new simplex is taken as x_1, x_2 and x_e and the process of reflection is again taken up. If, however, the expansion step is not a success, this point is not considered and we start reflection by considering x_1, x_2, x_r.

If the point x_r obtained as a result of reflection is worst than all the points on simplex except x_h, it does indicate that probably we have gone too far in that direction. In such cases, the simplex is contracted as follows

$$x_c = \beta x_h + (1-\beta)x_o \qquad ...(32)$$

where x_c is the vertex obtained from contraction and β is contraction coefficient $0 < \beta < 1$. Generally β is assumed 0.5. The method is assumed to have converged when the standard deviation of the function at the (n+1) vertices of the current simplex is less than prespecified tolerance (ϵ). Mathematically, if

$$Q = \left[\sum_{i=1}^{n+1} \frac{[f(x_i) - f(x_o)]^2}{n+1} \right]^{1/2} < \epsilon \qquad ...(33)$$

the procedure is terminated. The progress of a simplex procedure on a quadratic objective function is illustrated in figure 7.

2.5.2 Powell's method

It is a very powerful and simple method based upon search directions. This method is a method of conjugate directions. The conjugate is defined in the following paragraph.

Let A be an (n*n) symmetric positive definite matrix. A matrix is called positive definite if all the elements of main leading diagonal are positive and all leading minors are positive. If any one of them is negative, the matrix is called negative definite. Alternatively, a matrix A will be called positive definite if all the values of λ which satisfy following equation are positive:

$$| A - \lambda I | = 0 \qquad ...(34)$$

where I is the identity matrix. Now, two vectors, S^1 and S^2 are conjugate to each other if

$$S^{1T} A S^2 = 0 \qquad ...(35)$$

Now after familiarizing with the basic definitions, the computational procedure of Powell's method is being demonstrated with the help of figure 8.

Let there be a two variable function which is to be minimized. We start

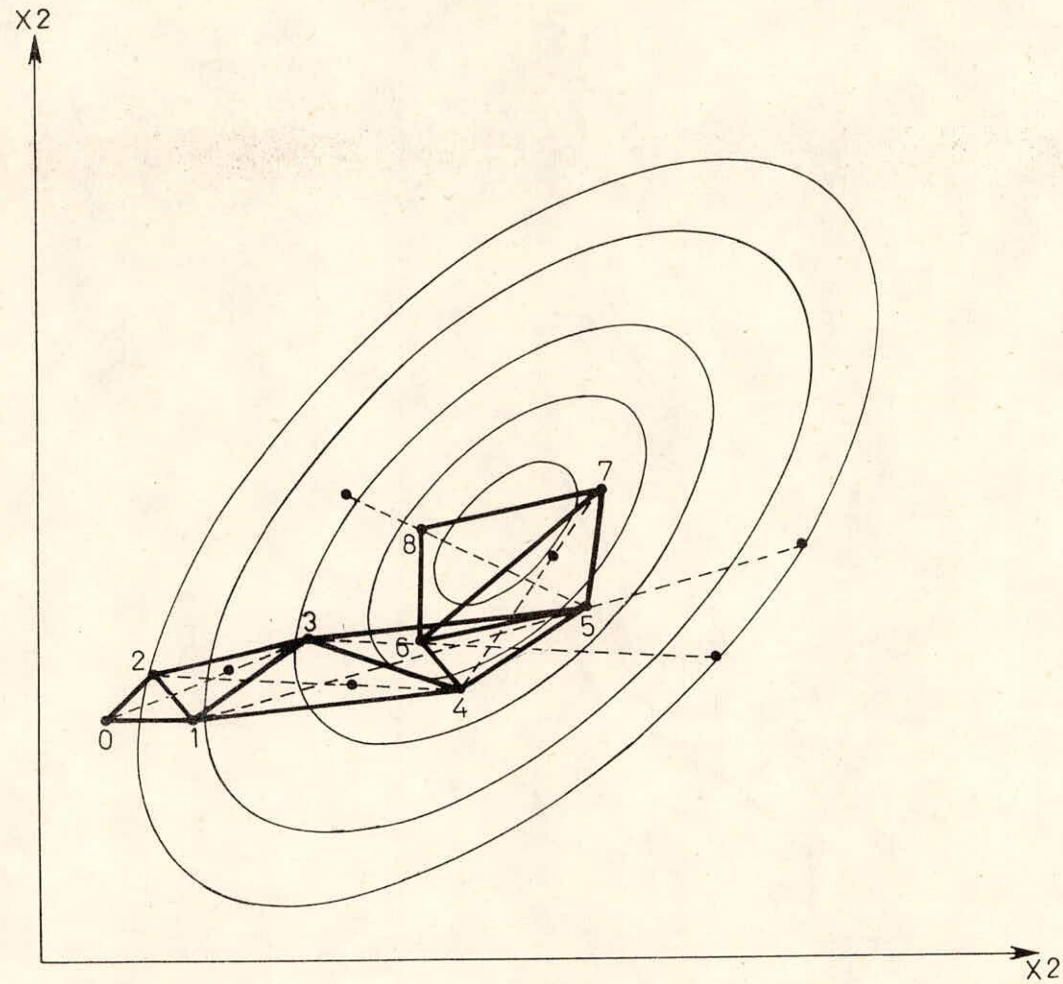

FIGURE - 7 THE SIMPLEX METHOD ILLUSTRATED ON A QUADRATIC FUNCTION

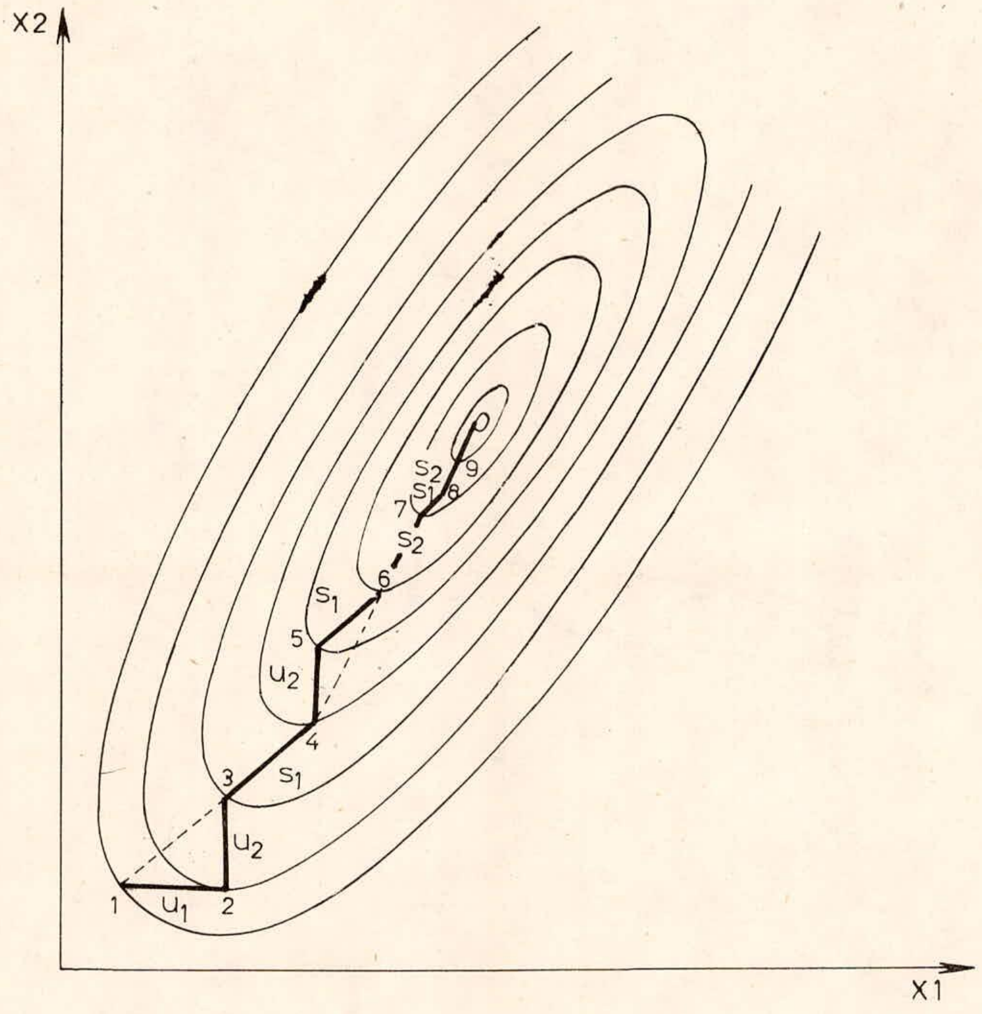

FIGURE - 8 PROGRESS OF POWELL'S METHOD(After Rao 1978) from the initial point 1. The function is first minimized along x_1 by moving (in positive x direction in this case), a finite step length, and the point 2 is obtained. Similarly, another step is taken in x_2 and the point 3 is obtained.

If 1 and 3 are joined, we get a pattern direction S_1. Now one of the coordinate directions, x_1 direction in the present case, is exchanged by S_1 direction and we minimize the function in this S_1 direction thereby yielding the point 4. From this point, we take a step in the direction x_2 which further minimizes the objective function and yields the point 5. Again another step is taken in the S_1 direction and point 6 is obtained. Now the second pattern direction S_2 is obtained by joining points 4 and 6 and a step is taken in this direction leading to point 7. This S_2 direction is now exchanged with the previously retained coordinate direction x_2 and minimization is done once along each pattern direction thereby yielding points 8 & 9 respectively.Now again we restart the minimization procedure by going in the coordinate directions as was done from point 1 onwards.

It can be proved that the Powell's method will minimize a quadratic function in a finite number of steps.

Besides above two methods, other important methods available in this category are Hooke and Jeeves method and Rosenbrock's Technique.

2.6 Gradient of a Function

A number of optimization methods for nonlinear optimization problems make use of gradient of the function. The gradient, denoted by ∇ ,of a function is a vector containing partial derivatives of the function with respect to each of the variables.

$$\nabla f = \begin{bmatrix} \partial f/ \partial x_1 \\ \partial f/ \partial x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \partial f/ \partial x_n \end{bmatrix} \quad \ldots (36)$$

The property of gradient which is most useful from optimization point of view is that if we move along the direction of gradient, the function value changes at the fastest rate. Thus since we are moving on the steepest slope, the convergence to the optimum is quickest. The methods which use this property are called steepest ascent (descent) methods. Among the popular methods in this cateogy are Fletcher-Reeves method, and Davidon-Fletcher-Powell method.

2.7 Constrained Nonlinear Optimization Problems

The general representation of this type of problems is

$$\text{Min } f(x)$$

$$\text{Subject to} \quad G_j(x) \leqslant 0 \qquad j = 1,2.....m \qquad ...(37)$$

The constraints can be handled either explicitly or implicitly. The methods in which explicit consideration of the constraints is made are termed direct methods, and the methods where constraints are considered implicitly are termed indirect methods. The indirect methods are more efficient and versatile then direct methods.

For the purpose of this report, the discussion will be limited to penalty function methods or SUMT, Rosen's Gradient Projection Method and the GRG Methods.

2.7.1 Penalty function methods

The idea behind penalty function approach is to convert the general constrained nonlinear programming problem into a sequence of unconstrained problems by incorporating the constraints into the objective function. These methods broadly come under SUMT or Sequential Unconstrained Minimization Techniques.

The penalty function methods can be further subdivided into two types: Interior Penalty Function methods and Exterior Penalty Function methods. Sometimes one more type, mixed methods is added to the list. In the interior penalty function methods, the objective is to retain the solution in the interior of the feasible region and then gradually proceed to the optimum. In contrast, in the exterior penalty function methods, the optimum is approached from outside of the feasible region. In the following article, only one of these, the interior penalty function method will be discussed in detail but before that one important aspect, i.e., the recognition of an optimum point is being discussed. The conditions, known as Kuhn-Tucker Conditions, are the necessary conditions to be satisfied at a relative optimum point.

2.7.2 Kuhn-Tucker Conditions

Let the objective function of the nonlinear optimization problem be F(x) and the constraints be $g_i(x)$, i=1,2...m. The objective function, which is a function of n variables, is differentiable and so are the constraints. Further, let $x \geqslant 0$. The optimal solution to this problem x^*,exists only if there exist variables $\lambda_1, \lambda_2 \dots\dots\dots \lambda_m$, such that the following Kuhn-Tucker conditions are satisfied.

1. If $x_j^* > 0$, then $\left. \frac{\partial F}{\partial x_i} + \sum_{i=1}^{m} \lambda \frac{\partial g_i}{\partial x_j} \right|_{x_j^*} = 0,$ ...(38)

$j=1,2,\dots\dots\dots\dots n$

2. If $x_j^* = 0$, then $\left. \frac{\partial F}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \right|_{x_j^*} > 0$ ...(39)

$j= 1, 2,\dots.n$

3. If $\lambda_i > 0$ then $g_i(x_1^*, x_2^*, \ldots x_n^*) = b_i$, $i=1,2,\ldots n$. ...(40)

4. If $\lambda_i = 0$ then $g_i(x_1, x_2 \ldots x_n) \leqslant b_i$, $i=1,2,\ldots m$ ...(41)

5. $x_j^* \geqslant 0$, $j = 1,2\ldots m$

6. $\lambda_i^* \geqslant 0$ $i = 1,2,\ldots m$

The first condition is necessary for an optimum if the optimal point is not at the boundary of the feasible region. Condition 2 is supplementary to the first condition when the optimum may lie on the boundary. In the third condition, the Lagragian multiplier for the constraint exists which means that the constraint is binding. Similarly, the fourth condition indicates that the corresponding Lagrangian multiplier is zero or the constraint is loose. Fifth and sixth conditions express the non negativity of the decision variables and the Lagrangian multipliers.

If the objective function and the constraints are convex then the Kuhn-Tucker Conditions are necessary as well as sufficient conditions for a global optimum.

2.7.3 Interior penalty function method

In this technique, a new function Φ is constructed which consists of the objective function and constraints. This function can be

$$\phi \quad (x,r_k) = f(x) - r_k \sum_{j=1}^{m} \frac{1}{g_j(x)} \qquad \ldots(42)$$

where $f(x)$ is the objective function, $g_j(x)$ are constraints, and r_k is penalty parameter. The above form of ϕ function is the simplest and other complex forms are also used. The value of the penalty term tends to increase as the constraint boundaries are approached; it is small in the interior of the feasible region. Further, the penalty term remains undefined if the point is infeasible. Thus, the method requires an initial feasible solution to start with. The behaviour of ϕ function is shown in figure 9.

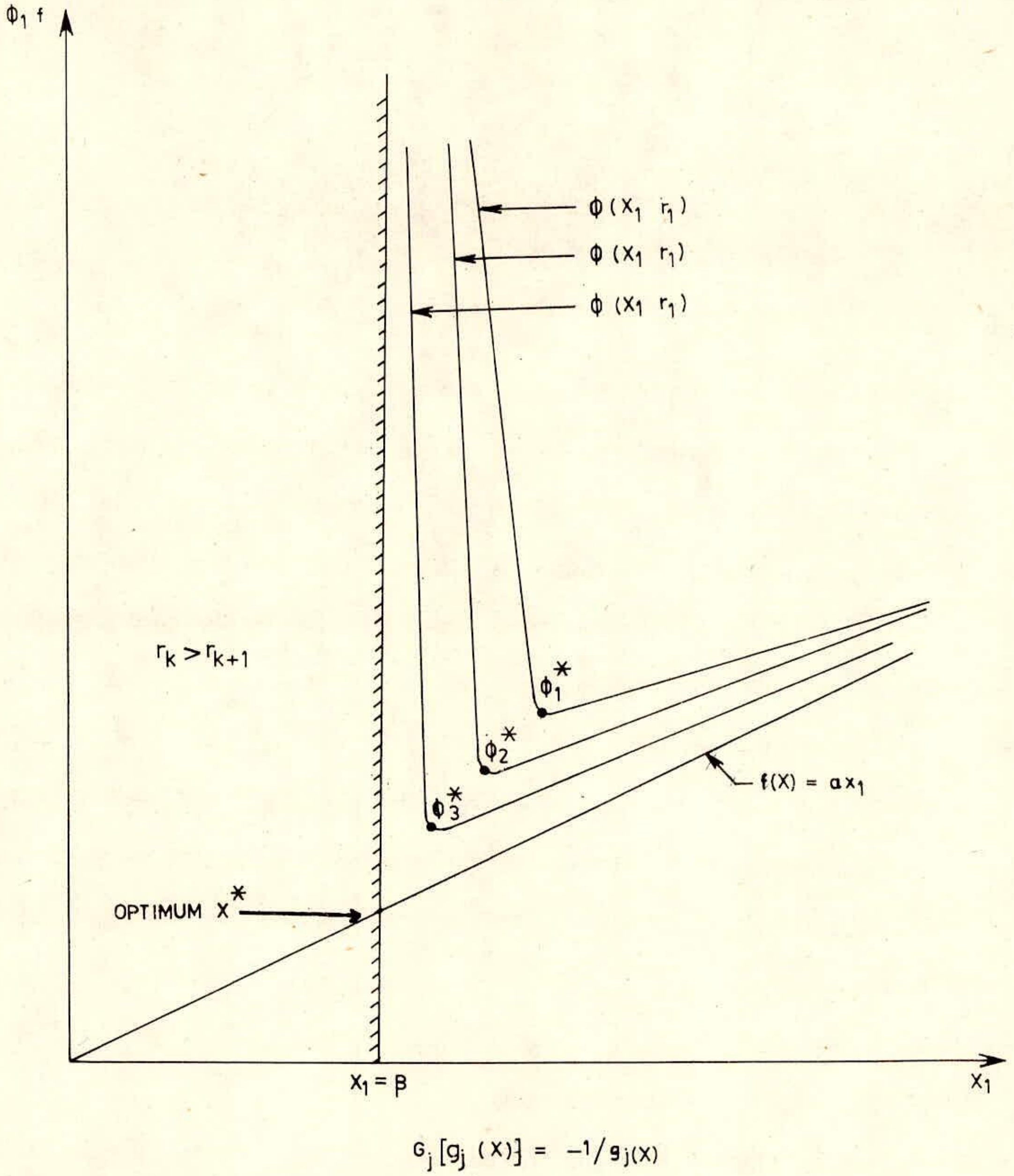

FIGURE - 9 ILLUSTRATION OF INTERIOR PENALTY FUNCTION METHOD

The penalty parameter r_k is greater than zero. Once its initial value is chosen, its value in subsequent iterations is always less than the previous value. Mathematically, we can express

$$r_{k+1} = C\, r_k \qquad \text{...(43)}$$

where $C < 1$

Rao (1979) gives following formula for choosing initial value of r:

$$r_1 \simeq 0.1 \text{ to } 1.0 \left[\frac{F(x_1)}{-\sum_{j=1}^{m} 1/g_j(x_1)} \right]$$

To solve a particular problem using this method, first the ϕ function is developed. Now after selecting an initial point, the function is minimized. For this purpose, any unconstrained minimization procedure can be used to obtain the optimal solution. If this solution is also optimal for the original problem, the process is terminated, otherwise a new value of the penalty parameter is obtained and the process is repeated. This time, the last optimal solution is used as the initial solution.

Several problems come up when solving a problem by penalty function method, e.g., determining an initial feasible solution and suitable value of penalty term and its subsequent reductions. Due to them, these methods are not very popular nowadays.

2.7.4 Rosen's gradient projection method

This method comes under the broad category of methods of feasible direction. In a way, the idea is same as in the unconstrained minimization alongwith incorporation of inequality constraints. First an initial feasible point is chosen. The directions in which further movement from this point is possible are of two types. Movement in some directions may violate one or more constraints and hence these directions may be infeasible. The other possibility is the movement in those directions which are feasible i.e., no constraints are violated consequent to the advancement. Among the feasible directions, the objective function will improve in certain directions which are called usuable feasible directions. This method is very effective if the constraints are linear though it can also be used to solve any nonlinear programming problem. In this method, the negative of the objective function gradient is used to find the usable feasible direction.

Let the optimization problem be of the form

$$\text{Min } F(X) \qquad \ldots(45)$$

Subject to

$$A\,X - b = 0 \qquad \ldots(46)$$

where X is n dimensional vector and the number of constraints is m. For simplicity, the constraints are assumed linear.

Suppose we are at a point X which is feasible and hence AX-b=0. Now we want to step to a nearby feasible point (X+dX) along the path of steepest descent. As this point is also feasible,

$$A(X+dX)-b=0 \qquad \ldots(47)$$

Hence

$$A\,dX=0 \qquad \ldots(48)$$

Now we want to move along a path ds to maximize the rate of change of the objective function. This can be expressed as

$$\frac{dF}{ds} = \left[\frac{\partial F}{\partial x}\right]^T \frac{dx}{ds} \qquad \ldots(49)$$

where

$$\partial F/\partial x = \nabla F(x) \qquad \ldots(50)$$

$$\frac{dx}{ds} = \left[\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_n}{ds}\right]^T \qquad \ldots(51)$$

The Lagrangian formed to maximize the above rate of change is

$$L\left(\frac{dx}{ds}, \lambda, \lambda_0\right) = \left(\frac{dF}{ds}\right)^T \frac{dx}{ds} + \lambda^T A \frac{dx}{ds}$$

$$+ \lambda_0 \left[1 - \left(\frac{dx}{ds}\right)^T \frac{dx}{ds}\right] \qquad \ldots(52)$$

Differentiating it

$$\nabla L \; dx/ds = \frac{\partial F}{\partial x} - 2\lambda 0 \frac{dx}{ds} + A^T\lambda = 0 \qquad \text{...(53)}$$

or

$$\frac{dx}{ds} = \left(\frac{\partial F}{\partial x} + A^T \lambda\right)/2\lambda 0 \qquad \text{...(54)}$$

and

$$\nabla L_\lambda = A \frac{dx}{ds} = 0 \qquad \text{... (55)}$$

$$\nabla L\lambda_0 = 1 - \left(\frac{dx}{ds}\right)^T \frac{dx}{ds} = 0 \qquad \text{...(56)}$$

Substituting for dx/ds in the equation (55) gives

$$A\left(\frac{\partial F}{\partial x} + A^T\lambda\right) /2\lambda_0 = 0 \qquad \text{...(57)}$$

Since the search direction is normalized according to equation (54), λ_0 will not be zero.

Hence

$$A\nabla F + AA^T\lambda = 0 \qquad \text{...(58)}$$

Or

$$\lambda = -(AA^T)^{-1} A \nabla F \qquad \text{...(59)}$$

Substituting this value in equation (54)

$$\frac{dx}{ds} = [\nabla F + A^T (-AA^T)^{-1} A \nabla F]/2\lambda_0$$

$$= [I - A^T(AA^T)^{-1} A] \nabla F/2\lambda_0$$

$$= P \nabla F/2\lambda_0 \qquad \text{... (60)}$$

where $P = I - A^T (AA^T)^{-1} A$ ... (61)

Matrix P is called the projection matrix and λ_0 can be regarded as the scaling factor. The matrix P projects the vector ∇F on the subspace defined by the constraint set. It is assumed that the constraints are independent so that columns of matrix A are linearly independent and hence (AA^T) is nonsingular.

Starting from initial point x_i, we move to the new point x_{i+1} along the direction S(=dx/ds) as

$$x_{i+1} = x_i + \lambda s \qquad \text{...(62)}$$

where λ is the step length. The algorithm is terminated when optimality is satisfied.

As mentioned earlier, this particular technique is not very effective in case the constraints are nonlinear in nature. The most popular technique of solving general nonlinear programming problems perhaps is a technique based on reduced gradient concept. It is called Generalized Reduced Gradient (GRG) technique. The computational requirements for GRG are significantly less than the other methods. For example, it has been reported that the ratio of computational requirement for GRG and interior penalty function method is around one-sixth.

2.7.5 Generalized reduced gradient method

The Generalized Reduced Gradient(GRG) technique is an extension of a previous technique proposed by Wolfe to solve a nonlinear optimization problem in presence of nonlinear constraints.

Let the problem be of the form

$$\text{Min } F(X) \qquad \ldots(63)$$

Subject to

$$g_i(X) = 0 \quad i=1,\ldots\ldots neq \qquad \ldots(64)$$

$$0 \leqslant g_i(X) \leqslant u_{n+i} \quad i= neq+1\ldots\ldots m \qquad \ldots(65)$$

$$l_i \leqslant x_i \leqslant u_i \quad i=1,\ldots\ldots n \qquad \ldots(66)$$

Here X is a vector of n variables, neq(which may be zero also) is the number of equality constraints, and l and u represent lower and upper bounds respectively. The functions F and g are assumed differentiable.

As a first step, the problem is converted to the following form by adding slack variables $x_{n+1} \ldots x_{n+m}$

$$\text{Min } F(X) \qquad \ldots(67)$$

Subject to

$$g_i(X) - x_{n+1} = 0, \; i=1,\ldots\ldots m \qquad \ldots(68)$$

$$l_i \leq u_i \leq 0, \quad i=n+1,\ldots\ldots n+m$$

$$l_i = u_i = 0; \; i = n+1,\ldots.n+neq$$

$$l_i = 0; \; i = n+neq + 1,\ldots.n+m$$

The last two equations represent bounds for the slack variables. The vector of natural variables can be split into two partitions, basic and non-basic (similar to what is done in Simplex method for linear programming). Let X_B be the vector of nb basic variables and X_N be the vector of (n-nb) nonbasic variables. The binding constraints can be written as

$$g(X_B, X_N) = 0 \qquad \ldots(69)$$

where g is the vector of nb binding constraints. The basic variables must be chosen such that the nb * nb basis matrix is nonsingular at a point $\bar{x}$ (which satisfies all the constraints). The binding constraints given by equation (69) may be solved for X_B in terms of X_N valid for all points in the neighbourhood of $\bar{x}$. Thus the objective function can be expressed as a function of X_N only and the problem reduces(in the neighbourhood of $\bar{x}$) to the following problem:

$$\text{Min } F(X_N) \qquad \ldots(70)$$

Subject to $$l \leq X_N \leq u \qquad \ldots(71)$$

The function $F(X_N)$ is called the reduced objective. Its gradient, denoted by $\nabla F(X_N)$, is called the reduced gradient. In GRG methods, a sequence of reduced problems is solved by using a gradient search method. The computational scheme is being described below. For the sake of simplicity, an example of two variable objective function is given first. Let the problem be

$$\text{Min } f(x_1, x_2) \qquad \ldots(72)$$

Subject to $$h(x_1, x_2) = 0 \qquad \ldots(73)$$

The objective function and constraints are assumed differentiable

$$d\, f(x) \; \frac{\partial f(x)}{\partial x_1} dx_1 + \frac{\partial f(x)}{\partial x_2} dx_2 \qquad \ldots(74)$$

$$d\, h(x) = \frac{\partial h(x)}{\partial x_1} dx_1 + \frac{\partial h(x)}{\partial x_2} dx_2 \qquad \ldots(75)$$

Since the constraint has to be satisfied at a feasible point

$$d\, h(x) = 0 \qquad \ldots(76)$$

or

$$\frac{\partial\, h(x)}{\partial\, x_1}\, dx_1 + \frac{\partial\, h(x)}{\partial\, x_2}\, dx_2 = 0 \qquad \ldots(77)$$

$$dx_2 = -\frac{\partial\, h(x)/\,\partial\, x_1}{\partial h(x)/\partial\, x_2}\, dx_1 \qquad \ldots(78)$$

Substituting for dx_2

$$d\, f(x) = \left[\frac{\partial f(x)}{\partial x_1} - \frac{\partial f(x)}{\partial\, x_2} \quad \frac{\partial\, h(x)/\ \partial x_i}{\partial\, h(x)/\partial\, x_2} \right] dx_1 \qquad \ldots(79)$$

or

$$\frac{df(x)}{dx_1} = \frac{\partial\, f(x)}{\partial x_1} - \frac{\partial\, f(x)}{\partial x_2} \left[\frac{\partial h(x)}{\partial\, x_2} \right]^{-1} \frac{\partial\, h(x)}{\partial\, x_1} \qquad \ldots(80)$$

The quantity $d\ f(x)/\ dx_1$ is the reduced gradient. To generalize, the differential of a function, $F(X)$, where X is a vector of variables which has been split in basic variables X_B and non basic variables X_N, is given by

$$d\, F(X)/d\, X_N = {}_{\nabla} X_N^{\,T}\ F + \nabla^T X \quad \frac{F d X_B}{d\ X_N} \qquad \ldots(81)$$

Similarly, form the constraint equation, we get

$$\frac{dh}{dX_N} = \frac{dh}{dX_N} + \frac{\partial h}{\partial X_B} * \frac{dX_B}{dX_N} = 0 \qquad \ldots(82)$$

or

$$\frac{dX_B}{dX_N} = -\left(\frac{\partial h}{\partial X_B}\right)^{-1} \frac{\partial h}{\partial X_N} \qquad \ldots(83)$$

Substituting from above equation in the equation (81) we get the following expression for the generalized reduced gradient

$$\frac{df(X)}{dX_N} = \nabla^T X_N\, F - \nabla^T X_B\ F\left(\frac{\partial\, h}{\partial\, X_B}\right)^{-1} \left(\frac{\partial h}{\partial\, X_N}\right) \qquad \ldots(84)$$

Now a search direction d is formed and a one dimensional search is initiated to solve the problem

$$\min_{\alpha > 0} F(\bar{x} + \alpha d) \quad ...(85)$$

where $\bar{x}$ is point which satisfies the constraints ,λ and α can be interpreted as a step length. $\bar{x}$ will be the initial point for first iteration. The search is terminated if an optimal point is found or if the algorithm does not coverge. It may also terminate if the algorithm converges at a point where one or more constraints are violated or the bounds on basic variables are not satisfied.

In the area of nonlinear programming the GRG methods are perhaps most versatile and efficient. Efficient programs have been developed (though not widely available so far) which use this technique. Mainly due to non availability of programs, the use of nonlinear programming in solving the problems of water resources systems is somewhat limited.

2.8 Dynamic Programming

Dynamic programming is defined as an enumerative technique which can be used to obtain optimal solution to a variety of problems. The objective function need not be convex. It has also been applied to non-sequential problems. Some of its advantages are:

(i) It can find the optimal solution even when the feasible region is not convex.

(ii) The solution can be found in cases where the variables are not continuous. Several problems in system design can not be realistically and effectively formulated using the concept of continuous variables.

(iii) It also provides solution to a number of problems where classic methods of calculus fail.

Dynamic programming was developed by Richard Bellman. Its entire theory is based upon Bellman's principle of optimality. The principle can be stated as follows (Ref. Fig.10): Given an optimal trajectory (I-II) from point A to C,

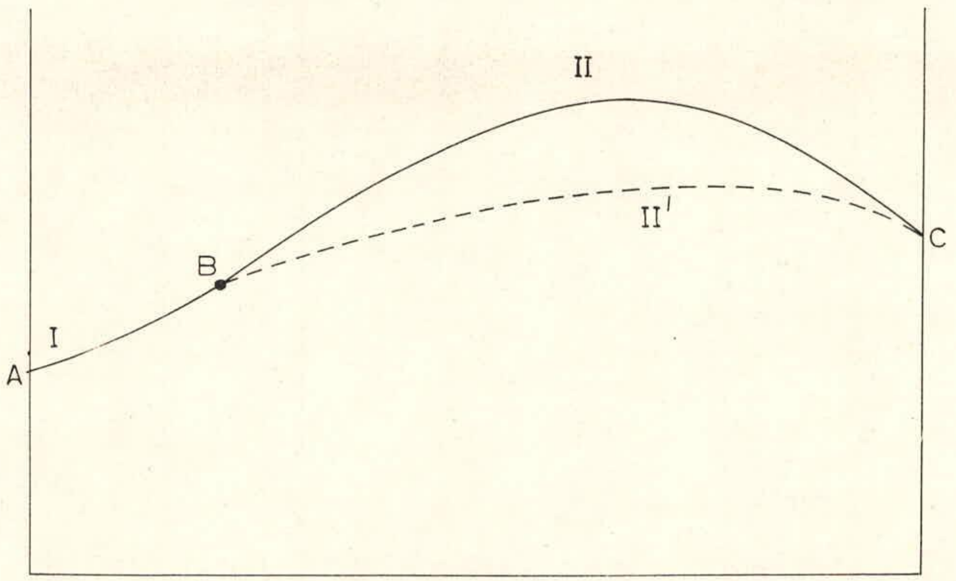

FIGURE 10 - ILLUSTRATION OF BELLMAN'S PRINCIPLE OF OPTIMALITY

the portion of this trajectory from any intermediate point B to point C (II) has to be the optimal in the interval B to C. The proof can be immediately obtained by contradiction. In the dynamic programming formulation, the dynamic behaviour of the system is expressed in terms of three variables; i) State variaobles which describes the system, ii) control variables which represent the decision or control applied and influence the process by affecting the state variables in some prescribed fashion, and iii) the stage variables which determine the order in which events occur in the system. Generally, the stage variable is taken to be time. The system is described by a set of equations called system equations which describe how the stage variables at stage t+1 are related to those at stage t, on application of control variables during stage t to t+1. It can be written in a general form as

$$x(t+1) = g\,[\,x(t),\, u(t),\, t] \qquad \ldots(86)$$

where

x = vector of state variables

u = control vector

t = stage variable

Some finite integration formula must be used to approximately represent the differential equation on a digital computer. The simplest can be

$$x(t + \delta t) = x(t) + g\,[x(t), u(t), t]\,\delta t \qquad \ldots(87)$$

where δt= time increment over which control $u(t)$ is applied.

The performance criterion can be taken to be a reward function which is to be maximized or else a cost function which is to be minimized. The approximate finite formula for cost function can be in the form

$$J = l\,[\,x(t), u(t),\, t]\,\, \delta t + [x(t_f), t_f] \qquad \ldots(88)$$

where

l = scaler functional for cost per unit time ,

ψ = scaler functional for final cost , and

t_f = final time.

The constraints can be expressed as

$$x \varepsilon X(t)$$

$$u \varepsilon U(x,t)$$

where X is the set of admissible states which can vary with time and U is the set of admissible controls which can vary with both X and t. The principle of optimality is applied to this problem to yield an iterative functional equation to determine minimum cost that is incurred in going to the final time from the present time. The minimum cost at a given state x(t) at present time t is found by minimizing, using the choices of the present control u(t), sum of cost over next time interval and minimum cost of going to final time from resulting next state. Mathematically it can be expressed as

$$I(x,t) = \min_{u \varepsilon U} \; l\,[x(t),u(t),t]\delta t + I\,[x(t) + f(x(t),u(t),t)\,\delta t, t+\delta t] \qquad \ldots(89)$$

The minimum cost function for all x and t can be evaluated by iteratively solving this equation with the boundary condition

$$I(x, t_f) = \psi(x, t_f) \qquad \ldots(90)$$

Thus, in condensed form, the optimization problem can be stated as :

Given - A system described by equation(86)

-constraints that $x \varepsilon X(t)$, $u \varepsilon U(x,t)$

- An initial state x(o)

Find-the control sequence u(o),.. u(t) that minimizes J while satisfying constraints.

The problem can be solved by enumeration. At the given state x(o) every admissible control $u \varepsilon U$ is applied and for each of them, next state is computed. In general, when a set of states x(t) has been defined by the procedure, a new set of state x(t+1) is defined by applying all $u \varepsilon U$ at all of the x (t) by

$$x(t+1) = g\,[x(t),u(t),t] \qquad ...(91)$$

The cost of admissible states is computed. This continues until the final value of time is reached.

Now we trace out all trajectories in state space that do not violate the constraints and that begin at x(o) and end at $t=t_f$. These trajectories form a tree beginning at x(o), and expanding as t increases. One such tree is illustrated in Fig.11. It is possible to handle a large variety of constraints by this method and incidently,they serve a useful purpose by reducing the number of trajectories that must be considered. The minimum cost is evaluated by comparing cost for the admissible states and choosing the minimum value. The optimal control sequence and optimal trajectory in state space are determined by tracing back along the path that led to this minimum value of cost. This procedure always leads to an absolute minimum rather than a relative minimum.

Inspite of all its advantages, the enumeration method leads to one major computational difficulty. Although constraints do reduce the number of trajectories , they increase exponentially as the computations move forward in time. Hence complex problems become unmanageable due to large computer memory requirements.This is what Bellman has called 'the curse of dimensionality'. Few of the several modified procedures that have been developed to overcome the curse of dimensionality are being discussed here.

2.8.1 State increment dynamic programming

The optimization problem to which state increment dynamic programming technique(SIDP) is applied can be conveniently expressed in the continuous time

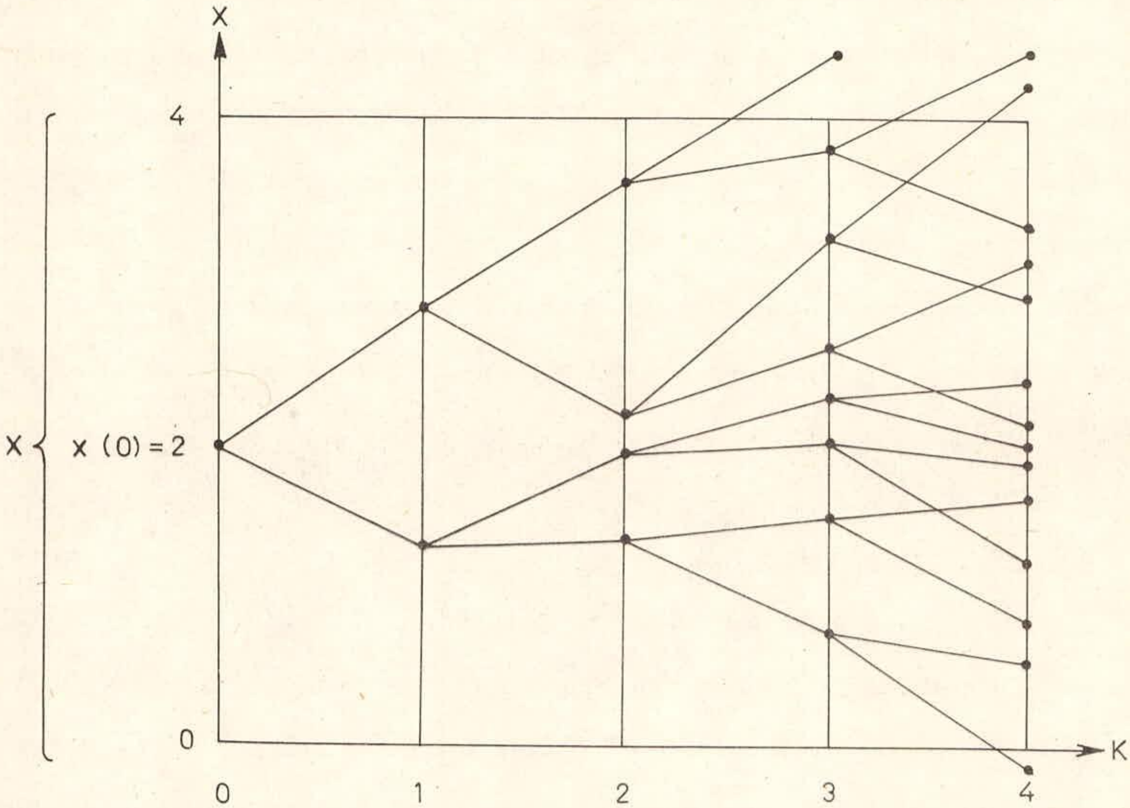

FIGURE - 11 TREE GENERATED BY ENUMERATION scale. The system equations become a set of nonlinear time varying differential equations :

$$dx/dt = f\ (x,u,\ t) \qquad ...(92)$$

where f = a vector functional.

A fundamental difference between state increment dynamic programming and the conventional procedure lies in the method for determining δt, the time interval over which a given control is applied. In conventional dynamic programming, the total time interval over which the optimization is performed is quantified into uniform increments, Δt, and the optimal control is computed only at these quantified values of t. On the other hand, in state increment dynamic programming, this time period is determined as the minimum time interval required for any one of the n state variables to change by one increment. Thus

$$\delta t = \min_{i=1,n} \left| \frac{\Delta x_i}{f_i(x,u,t)} \right| \qquad ...(93)$$

where Δx_i = change in the ith state variable.

In general δt is determined by applying control until the trajectory reaches a n dimensional hypercube centered at the current state and with side of $2\Delta x_i$ along the ith coordinate axis. As a resualt the next state is always close to the present state. Thus it is necessary to store values of minimum cost function at only those quantified states which are nearer to the present state. This enables the state increment dynamic programming to reduce the computational requirement from those of conventional dynamic programming. Further, an overall significant saving in high speed memory requirement(the amount of data to be stored in order to apply the iterative functional equations at a specified $x \in X$ and $t_o < t < t_f$) can be achieved by efficiently processing data. This is achieved by utilising the concept of blocks which are defined by partitioning the (n+1) dimensional space (containing the n state variables and time) into rectangular sub-units. The concept and methodology of state increment dynamic programming has been discussed in detail by Larson (1968).

2.8.2 Discrete Differential Dynamic Programming

Discrete Differential Dynamic Programming (DDDP) as proposed by Heidari et.al.(1971) is a computational technique parallel to state increment dynamic programming. This method is an improved form of discrete dynamic programming.

This meothod starts with a trial trajectory denoted by x'(t), satisfying a specific set of initial and final conditions. The sequence of controls applied to obtain the trial trajectory is called trial policy. Total returns from this trial policy and trial trajectory over entire time horizon are obtained. Now a set of incremental state vectors are defined as

$$\Delta\, x_i(t) = [\, \delta x_{i1}(t)\ \delta\, x_{i2}(t)\ \ldots\ \delta x_{ij}(t) \ldots\ \delta\, x_{in}(t)\,] \qquad \ldots(94)$$

where jth component can take any one value σ t, t=1,2,...k,from a set of assumed incremental values of the state domain. When added to the trial trajectory at a stage, these vectors form an n-dimensional sub domain:

$$D(t) = x'(t) + \Delta x_i\ (t),\ i = 1,2\ldots K \qquad \ldots(95)$$

Here one value of σ t has to be zero because the trial trajectory is always in the subdomain. All D(t) t=1...K together are called a 'Corridor'. A corridor is used as a set of admissible states and optimization is performed constrained to these states. This leads to a new value of return. If this value is greater than the value obtained in earlier iteration (less in case of minimization problem) then the procedure is repeated from the beginning until the difference is less than a preassigned tolerance limit. For the next iteration, the optimal policy from the previous iteration becomes the trial policy and a new corridor is formed around it. The width of the corridor goes on reducing with each iteration to have a finer policy. An illustration of this method is given in figure 12.

The suggested method has been found to be particularly helpful in case of invertible systems(A system is said to be invertible if the order of the state vector is equal to the order of the decision vector).

2.9.3 Stochastic dynamic programming

Stochastic programming deals with situations where some or all the parameters of the optimization problem are described by stochastic variables. This condition is frequently encountered in reservoir operation problems because future inflows to reservoir are always stochastic in nature. Depending upon the nature of the programming problem, it is called a stochastic linear programming, stochastic nonlinear programming or a stochastic dynamic programming technique. Out of these three, stochastic dynamic programming has been very popular and there has been a proliferation of papers on it during the recent times. The procedure is briefly described here.

Consider a stochastic return function

$$R_t = R_t\ (x\ (t+1)\ ,\ u(t),\ y_t) \qquad \ldots(96)$$

where y_t is a random variable which will be absent in the case of a deterministic return function. Let y_t be discrete with a probability mass function of $p_t(y_t)$. The random variables y_1, y_2 .....y_n are assumed to be statistically independent. For a fixed value of x (t+1) and u(t), we would expect on an average, a return of

$$\bar{R}_t\ (x(t+1),u(t)= \sum_{y_t} P_t(y_t)R_t(x(t+1),u(t),y_t) \qquad \ldots(97)$$

An important property of the expected return is that it statistically represents an estimate of the average return from any one trial even through it may not be possible to receive the amount $\bar{R}_t$ (x(t+1), u(t))in practice. The objective function to be optimized is given by the sum of individual stage returns:

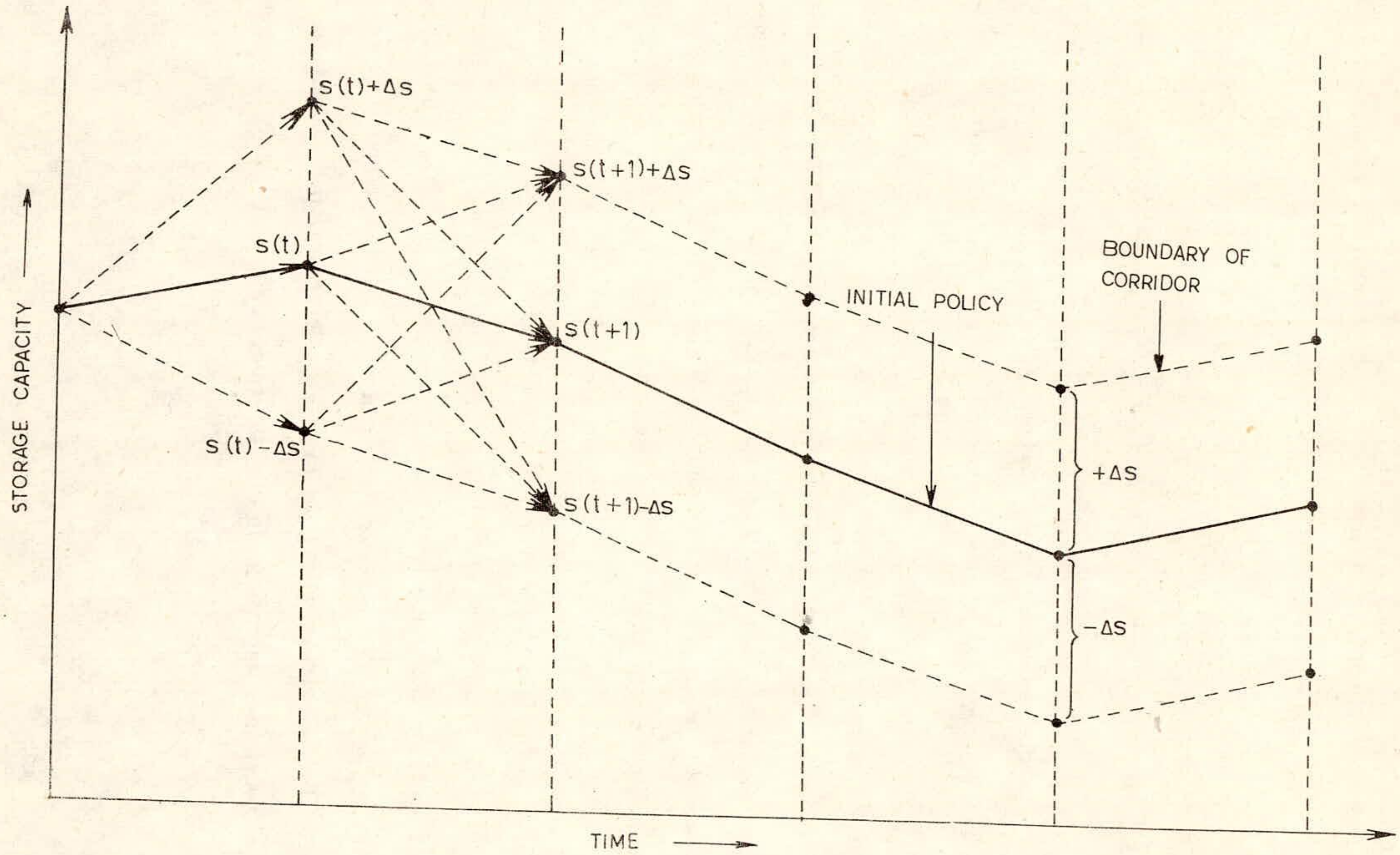

FIGURE- 12 DISCRETE DIFFERENTIAL DYNAMIC PROGRAMMING

$$F\ (u(1),u(2)...u(n)) = \sum_{t=1}^{n} R_t\ (x(t+1),u(t),y_t) \qquad ...(98)$$

The following stochastic recurrence can be readily derived, Rao (1979):

$$F^*[x(t+1)] = \max_{u(t)} \sum_{y_t} p_t(y_t) Q_t[\ x(t+1),u(t),y_t] \qquad ...(99)$$

$$1 \leqslant t \leqslant n$$

where

$$Q_t\ [x(t+1),u(t),y_t] = R_t\ [x(t+1),u(t),y_t] + F^*_{t-1}\ [g(x(t+1),u(t),y_t)] \qquad ...(100)$$

$$2 \leqslant t \leqslant n$$

and $$Q_1[x(2),\ u(1),\ y_1] = R_1\ [x(2),u(1),y_1] \qquad ...(101)$$

$F^*[x(t+1)]$ denotes the maximum expected return as a function of the input state $x(t+1)$.

Stochastic dynamic programming yields an optimal policy which self stochastic except for the first optimal decision as pointed by Rao(1979). The remaining optimal decisions obtained in the form $u^*(t_{f-1})...u^*(2)$ by using the recurrence relations can not be expressed deterministically in terms of $u^*(t_f)$ until the random variables that precede them are revealed.

The development of above modified techniques has considerably reduced the computer memory requirements. This has resulted in a proliferation of studies which have used dynamic programming as solution technique. This is inspite of the fact that the technique itself is not tailored in such a fashion that generalized computer programs can be written. Still other advantages have outweighted this aspect. The efforts required to develop a program, particularly when it is used to assist the operation in the day-to-day management of the system, are worth it.

3.0 APPLICATIONS OF OPTIMIZATION MODELS TO PROBLEMS OF RESERVOIR OPERATION

3.1 Types of Models

Several types of models have been developed to analyse a reservoir system. The classification of models among different types is more of academic interest. Each model type has its own inherent advantages and limitations and there is no single model which can be applied for all problems associated with a water resources system.

All the available models, in general, can be classified as static or dynamic models depending upon assumptions about hydrologic and economic conditions. However, models which are static with respect to one process may be dynamic with respect to other processes.

Water resources system models can also be classified as deterministic if the streamflows are assumed known and as probabilistic or stochastic if only the probability distribution of the streamflows is known. Although the stochastic models are more complex and require more computation time, the information supplied by them is more useful for analysis and due to this reason, they are in vogue nowadays.

The most important classification, however, from the point of view of this study is whether a model is a simulation model or an optimization model.

3.1.1 Simulation models

Simulation, which is essentially a search procedure, is one of the most widely used approaches for evaluating the performance of alternative decisions concerning a water resources system. The model, as developed for a digital computer ,consists of a series of mathematical instructions and relations representing the inherent characteristics, behaviour, and response of a particular system. This sequence of instructions, when carried out using historical or synthetic data as input ,simulates the operation of that system. The output from the model can then be used to analyse and evaluate the performance of the system.

3.1.2 Optimization Models

These models optimize the decision maker's choice which is expressed by an objective function. The choice is further subjected to a set of constraints which arise due to physical limitations, laws of nature, and resource availability etc. The solution is given in terms of specific values of decision variables which will give optimum value of objective function. Further, post optimality analysis can also be performed which will indicate how much sensitive optimal solution is to changes in decision variables and constraints.

3.2 Comparison of Simulation and Optimization Models

Using a simulation model it is possible to immediately answer the question-what if? Popularity of simulation models is due to their flexibility and the ease with which they may be developed and used. Even for a layman, their output is easy to understand. They have been found to be effective for evaluating alternative configurations of river basin development,water allocation alternatives, and feasible operating policies. The models can be very complex and realistic too. The decision maker's choice can be easily incorporated in the model. These models, however, have few limitations. One of them is that these models do not provide optimal results . If the basin offers a large number of alternatives, the problem of choosing best among them or selecting a set of alternatives which are to be further investigated in detail becomes uncontrollable in term of computation time and efforts required to screen the results. Secondly, although simulation provides great freedom to test diferent combinations of structures and target outputs, it does not give sufficient freedom to test different operating procedures. A major change in operating procedure requires re-writing, and testing a new programme. Finally, sampling in multidimensional space also makes the solutions questionable. Problems regarding adequacy, suitability, and representativeness of a sample are also not fully understood at present.

The optimization methods try to find out a set of decision variables such that the objective function is optimized (i.e.,either maximized or minimized). Due to the very nature of these models, they are very much helpful in initial screening of several alternatives of water resources development. The models become particularly helpful in studying the operation strategies of reservoirs. The decision making choice can be explicitly expressed through objective function. Versatile programs can be written which can take widely varying nature of objective functions. Unlike simulation models, only single run of the program is required to obtain the optimal solution. Post optimality analysis, also called sensitivity analysis is an important and useful part of an optimization model. Here it can be determined as to how much sensitive an optimal solution is to changes in the value of decision variables and how it changes by loosening or tightening of a particular constraint by a fixed amount. Nowadays very efficient packages of optimization programs are available which have made their use more or less universal. For complex and widely varying problems of reservoir operation, these models have taken a big lead over other types.

3.3 Choice of Objective Function

The choice of objective function is a very important decision for an optimization formulation. In the operation of a reservoir, a large number of alternatives are available to the decision maker. The output from each alternative can be measured in terms of say, a common monetary unit or crop yield or quantum of energy generated etc. Now to choose a particular alternative, it is necessary to order them in terms of attainment of the objective of the planner. The criterian used in this ordering or ranking is called the objective function.

Starting with the main objective of a water resource development as increasing national income, Maass et al (1962) introduced the notion of economic efficiency. According to them, a project will be called economically efficient if no alternative design would make any member of the community better off without making some other members worse off. To evaluate the economic efficiency, they introduced the willingness to pay criteria. The willingness of the people, affected by a particular project, to pay for it in terms of zero design (i.e., no project at all) can be measured and can be used to rank the projects. However, these types of criteria can be useful only in the design stage to screen the alternatives and are not helpful in the operation stage of a project. Nevertheless, it can be shown that all the objectives are reducible to economic efficiency objective.

In spite of the importanceof choice of objective function for a problem, detailed guidelines are not available for its selection for a particular problem. Usually, the choice of the objective function is governed by the nature of the problem and also the computational facilities available. For example, may times a linear objective function is chosen because it reduces the computational efforts significantly and moreover, efficient computer programs are widely available for linear programing problems. One way of converting nonlinear objective function to a linear one is piecewise linearization. In this approach, the function is divided into a number of segments such that it behaves linearly in each segment. The number of segments depends upon the nature of objective function, permitted distortions as a result of this process and also the computational facility(e.g.time) available. A finer division will, no doubt be more precise but will lead to more processor time.

The nature of the problem is another factor upon which the design of objective function depends very much. If the problem is of short term operation, the aim may be to evolve a policy which meets the targets as closely as possible. For example, for problems of flood control operation, objective may be to minimize the flood damage, or it may be to minimize the flows which are greater than the safe carrying capacity of the channel. Similarly, for conservation operation, the aim may be to minimize the deviations from the long term targets. Another interesting problem is the multipurpose operation of a reservoir. In this case the objective function should be designed such that all the purposes are given appropriate weight.

For a long term operation problem, the aim is more to fix the targets or the maximum attainable level of power or water. This may be in the form of maximization of firm power or firm water etc.

If the problem is concerned with the operation of a system of reservoirs, the concept of zones is frequently used. According to this concept, the aim is to ensure that all the reservoirs of the system are in the same storage zone at a given time. To achieve this, the demands are suitably divided among the reservoirs. This concept is similar to the concept of keeping all the reservoirs at the same index level.

3.4 Application of Optimization Techniques to Reservoir Operation Problems

A large number of studies have been made on reservoir operation problems. The reasons for this proliferation of studies are very clear. More stringent demands are being placed by society on existing water resources which require better management for higher degree of satisficing. Furthermore, it has been very well established that benefits derived from the joint operation of system of reservoirs substantially exceed the sum of benefits obtained from independent operation of each of the reservoirs.

Out of various optimization techniques discussed, two have been most commonly used for reservoir operation problems. These are linear programming and dynamic programming. Mostly dynamic programing has been used in case objective function is nonlinear. The user of integer programming is mostly for capacity expansion problems. Stochastic programming in conjunction with LP or DP has also been widely used. For the purpose of this study the works being reviewed are being classified into two broad categories:

(1) Linear programming techniques, and

(2) Dynamic programming techniques.

3.5 Linear Programming Techniques

The applications of linear programming to water resources systems in general and to reservoir operation in particular have been far and wide. But before discussing them,we will discuss a simple problem formulation.

Let us consider that a reservoir is located at a particular site. The dead storage capacity of the reservoir is S_{min} and the maximum capacity is S_{max} For the most critical year in past, monthly inflow data for the year is available and it is required to find out the annual firm water yield of the reservoir. The monthly distribution for this yield is also available, i.e., if the annual yield is X than its distribution among 12 months $\alpha_1 X, \alpha_2 X, \alpha_3 X.....\alpha_{12} X$ is known or in other words α_i, i = 1,12 are known.

If the water releasedfor the i^{th} month is R_i than it is essential that this value must be greater than or equal to $\alpha_i X$ for that month. This can be expressed mathematically as

$$R_i \geq \alpha_i X \qquad ...(102)$$

This condition must be satisfied for every month. Hence there will be 12 such constraints

$$R_i \geq \alpha_i X \quad i = 1,.....12 \qquad ...(103)$$

If the storage in any month is represented by S_i and inflow by I_i then according to continuity equation, the sum of initial storage and inflow for the i^{th} month less release during that month will be equal to the initial storage for next month. All losses are being ignored for the sake of present example. Further, there will be twelve such constraints, i.e.,

$$S_i + I_i - R_i = S_{i+1} \quad i=1,.....12 \qquad ...(104)$$

Apart from these constraints, from the physical point of view, the storage cannot exceed the maximum storage any time. This will also give rise to twelve constraints of the form

$$S_i \leq S_{max} \quad i=1,.....12 \qquad ...(105)$$

Similarly, it must also be ensured that the storage never falls below the minimum permissible value, or

$$S_i \geq S_{min} \quad i=1,.....12 \qquad ...(106)$$

The important thing left is the design of objective function. As the aim here is to maximize the firm yield, it can be put as

$$\text{Max } Z = X \qquad ...(107)$$

The entire problem can be put here as

$$\text{Max } Z = X \qquad ...(108)$$

Subject to $\quad R_i > \alpha_i X \quad i=1,.....12.$

$$S_i+I_i-R_i = S_{i+1} \quad i=1,.....12$$

$$S_i \leq S_{max} \quad i=1......12$$

$$S_i \geq S_{min} \quad i=1,.....12$$

$$R_i \geq 0 \qquad i=1,.....12$$

The problem has 48 constraints. It can be easily solved using a standard LP package. But the number of constraints is rather large.

It may also be mentioned that all the losses have been ignored in the present example. If evaporation losses are to be considered then the area capacity curve has to be used which is not linear in nature. One way out is to divide this curve in a number of segments and then linearize it in between two segments. This process is known as piecewise linearization. The accuracy, of course, depends upon the number of segments into which the entire region is divided.

In the above example, only one year of data(12 months), was used. However, in practice this may never be the case. Thus, for example, if data of 10 years is used then the number of constraints will increase to 480. This undoubtedly is a large number and it increases rapidly as the number of years is increased. If the analysis is carried using yearly data rather than monthly then the number of constraints will sharply reduce but this is not very much advisable. One remedy is to explicitly treat the stochasticity of the inflows by using the technique of stochastic linear programming. In this technique, an appropriate probability distribution is assumed(or fit) for the inflows. Now we take a particular reliability level, say 95%, and in the continuity equation, use that particular value of inflow which is exceeded with 0.95 probability. The modified constraint is written as

$$S_i + I_i(p) - R_i = S_{i+1} \qquad i=1,.....12 \qquad ...(110)$$

where $I_i(p)$ is the inflow value which is exceeded with a probability of p in the month i. This formulation will reduce the number of constraints to 48 with the added advantage that the stochasticity of inflows has been taken into account. Further, the sensitivity of the reliability parameter can also be studied.

This formulation, although convenient to use, has been severly criticised. One criticism is about the implicit assumption that the critical flow will occur in each month of the critical year. This assumption is not true in reality and it leads to conservative values. There is still a great controversy in the literature as to whether the error is really serious or not.

After explaining the formulation of the problem in linear programming framework,the attention is now focussed on reported applications of this technique. Among the first reported works using linear programming, the study carried out in the Harward water program is probably most significant. In this study, as reported by Maass et al (1962), a stochastic LP technique was used to find out the optimal reservoir operating policy.

The total inflow in a given period was assumed to be a stochastic variate. It may be pointed out here that as the inflow is discretized in finer or smaller units, the better is the representation of the actual condition. A too fine discretization will however, significantly increase the amount of computation. Similar arguments hold good while deciding about the number of time periods in which a year is divided for computational purposes. The number of time periods should be such that the streamflow and demand pattern is faithfully reproduced at no significant increase in the cost of computations.

In the above model, the initial state of the system was defined using two variables - inflow and storage. It was assumed that there is no serial correlation between the inflows in succeeding periods. The constraints included the continuity equations(one for each period) and constraints for limiting sotrage to the maximum reservoir capacity in each period. The decision variable for the problem was draft to be permitted from storage so that the expected value of benefit may be maximized. The objective function was to maximize the expected value of benefits over entire operating horizon, for all values of storage and release. The consistency of LP solution was checked using queuing theory.

Although this study was very successful, it had some limitations. The model developed could be used to study operation of a single reservoir only. Multiple purposes could be explored only if they could be combined into a single objective function. Serial correlation between inflows was also not permitted. The number of time periods was restricted to computer memory capacity which was quite small in those times. The computation of hydrolectric power benefits poses a problem in LP models. In the above model, it was assumed that the energy output is a function of end-of-the period storage and water released. If the analysis had to be done for flood control, the assumption was that average flows and flood damages have a good correlation. The computations for the above model which was applied to a hypothetical basin were performed on a IBM 700 computer.

The most important and yet most controversial application of the LP to problems of reservoir system has been in the framework of Linear Decision Rule(LDR). This rule, which was originally proposed by ReVelle et.al.(1969), can easily be applied both in deterministic and stochastic framework. In the LDR, it is assumed that the release is a linear function of storage, or

$$R_t = S_{t-1} - b_t \qquad \ldots (111)$$

Where R_t is the release during time period t, S_{t-1} is the storage at the end of period (t-1) and b_t is the decision rule parameter which optimizes the chosen objective function. The problem posed by ReVelle et al (1969) was to find the smallest reservoir which will deliver required flows under the physical constraints and equation(111). Rules, such as power rule, fractional rule etc. may perform better than LDR but the additional computational burden may not be worth while. The authors presented LDR in deterministic as well as chance constrained formulation. In the chance-constrained formulation, flows are known only with a certain probability. Similarly, the constraints also have probability values attached with them. They may be present to ensure that (i) the release must exceed a minimum value with a specified probability, (ii) the release must not exceed a maximum value with a specified probability, (iii) the storage must not go below a particular value with a specified possibility, and (iv) a minimum amount of freeboard must be available with a specified probability. Since its introduction, the LDR has been modified to consider evaporation losses, hydroelectric power benefits and extended for multi reservoir systems.

ReVelle and Gundelach (1975) presented a new LDR in which release is a function of current storage as well as past inflows:

$$R_t = S_{t-1} + \beta_t I_t - \beta_{t-1} I_{t-1} \cdots\cdots \beta_{t-k} I_{t-k} + b_t \qquad \ldots(112)$$

where β_t, $\beta_{t-1} \cdots\cdots \beta_{t-k}$ are constants to be determined. The authors pointed out that this form of LDR permits a smaller reservoir capacity than the original LDR. They used the condition of minimum variance of release to determine β weights.

Loucks and Dorfman(1975) pointed out that the LDR models give conservative results. The reason is that these models assume that critical flow and critical storage simultaneously occur in each period. In practice, their joint probability is negligible and this too depends upon the stochastic structure of the streamflows. They presented a LDR of the form:

$$R_t = S_t + \lambda_t I_t - b_t \qquad \ldots(113)$$

$$0 \leq \lambda t \leq 1$$

where λ_t is a coefficient. The choice $\lambda_t = 1$ gives the least conservative result while $\lambda_t = 0$ gives most conservative result.

Houck (1979) pointed out that the conservative nature of the LDR can be tackled by using conditioned cumulative streamflow distribution functions in the model. He presented multiple linear decision rule in which the release were conditioned on the previous two seasons streamflows and the storages on the streamflow of two previous months. The model can be easily formulated in other ways also choosing the number of events on which the releases and storages are to be conditioned by considering the accuracy of future streamflow predictions, modelling approximations, and available computational facilities.

In another framework of LDR, presented by Houck et al (1980), the hydroelectric energy generation and economic efficiency benefits were incorporated in the model. The economic efficiency was incorporated in their model as follows. The deterministic constraints, such as for storage and release which are bounded between upper and lower limits, were replaced by chance constraints for particular reliability level α . It was assumed that the targets represent long term benefits and the deviations represent short term losses. If a large number of probability levels are chosen such that they represent entire range from zero to one, the cumulative distribution function of deviations can be found out. In a flood control benefit case, the target free board represents the long term benefits. Further, the maximum possible freeboard equals the reservoir capacity and the minimum is zero. In objective function, the expected flood control benefits can be expressed as a function of the target, the probability level and the excess and deficit. As the long term benefit functions are concave and short term loss functions are convex, they can be easily incorporated in the model.

In case of hydroelectric power generation, the production function is nonlinear. It was assumed that the reservoir is operated such that during a time period, the volume remains within a small region with high probability. The relationship between storage and head was assumed linear within this small region. The values of expected head and release were used in the benefit function.

In the LDR models discussed so far, it has been assumed that the streamflows in successive periods are independent. It is well known that this assumption is not correct. The inflow time series has significant serial correlation which does influence the required reservoir capacity and release decisions. Joeres et.al (1981) presented a generalized formulation where correlation betweer the inflows was also considered. They presented these LDR's: independent rule where the correlation between the inflows in successive periods is zero or $\rho=0$, the predictive rule which assumes that $-1 < \rho < 1$, and the utopian rule or the case where perfect prediction is possible or $|\rho| = 1$

Houck and Dutta(1981) compared single -LDR with multiple-LDR. They showed that multiple-LDR is better than single LDR. But as the number of rules per season increases, the size of multiple-LDR also increases although they remain within the computational limits.

Loucks (1968) developed a stochastic linear programming model for a single reservoir optimization. The net inflows to the reservoir were assumed to follow a first order Markov chain. The first order transition probabilities were computed by observing the number of times the net inflow equalled j in period (t+1) after having been i in the period t, divided by the total number of transitions from i to all inflows j. These transition probabilities, when coupled with the reservoir operation policy, determine the probabilities of the transition of one storage volume to another. The objective function was minimization of sum of the expected squared deviations from the target reservoir volumes or discharges. Mathematically, the objective function was

$$\text{Min} \sum_{v,i,d,t} \alpha_t (v-v_t)^2 + (1-\alpha_t)(d-\sigma_t)^2 \, x_{vidt} \qquad \ldots(114)$$

where v_t and σ_t are target initial reservoir volume and discharges in period t, x_{vidt} is joint probability of beginning with a reservoir volume v, having an inflow i and discharging d in period t. α_t is a weighting factor expressing priority of reservoir volume to reservoir release in period t, such that $0 < \alpha_t < 1$

further $$\sum_{v,i,d} x_{vidt} = 1 \quad \text{for all } t \qquad \ldots(115)$$

The probability distributions of the resulting actual reservoir volumes and releases were obtained as solution in addition to optimal operating policies.

This approach has one drawback when applied to real life situations. A finer divisions of storages and inflows in discrete units is essential for sufficiently accurate representation of the system. This, however, leads to a considerably big transition matrix which also increases in size as the number of time periods for computations increases.

Houck and Cohon (1978) showed that this type of models belong to a broader category of stochastic nonlinear programming models. They also assumed that the inflows have a discrete Markovian structure. The formulation by Loucks(1968) was extended to a multipurpose multireservoir system. As the solution of nonlinear programming problems is very much difficult, the formulation was approximated by two LP's which iterated back-and-forth and converged to the optimum.

3.6 Dynamic Programming Techniques

As defined earlier, dynamic programming (DP) represents a multistage decision process. This technique has been extensively used for studying various aspects of water resources systems. One main reason of popularity of DP is that nonlinearity of benefits and stochasticity of inflows etc. can be easily incorporated in the DP formulation of a problem. Buras(1966) has given a detailed discussion on theory of DP and its applications to water resources problems.

A landmark in this direction was the application of DP to maximization of firm power from a two reservoir system. The firm power is defined as the maximum amount of power that can be delivered each year 100% of time according to some prescribed monthly distribution. The reservoirs in the system which was studied were in parallel. The maximum firm power that can be obtained by the operation of reservoirs is given by:

$$\max_{R(t)} \min_{t} \left[\frac{E_1(t) + E_2(t)}{B_j} \right] \qquad \ldots(116)$$

where $E_1(t)$ = amount of on-peak energy produced by first reservoir during time period t,

$E_2(t)$ = amount of on-peak energy produced by second reservoir during time period t,

B_j = a distribution coefficient for the j^{th} month, and

R(t) = vector of releases from first and second reservoir.

It was reported that from the point of view of computational efficiency, the increments to the state variables should be kept constant throughout any iteration and the increment size should be reduced as the iterations proceed. Youn g (1967) presented the reservoir operation problem in a control-theoretic framework. The technique which he has presented is called Monte-Carlo Dynamic Programming (MCDP). He assumed that the inflows follow a Gaussian lag-one Markov population. He generated 1000 traces of inflows and then the optimal storages and drafts were found for each of these using a forward looking deterministic dynamic programming algorithm. A least square regression was then performed for finding optimal operating policies. The loss function assumed was a single stage loss which depends on the release only. Young also used calculus of variation to obtain an analytic solution of a. continuous time version of the deterministic reservoir operation problem.

A technique, called Discrete Differential Dynamic Programming (DDDP) was proposed by Heidari et al (1971) to alleviate the problem of'curse of dimension ality'. The problem arises when the number of variables to be stored for a computation exceeds the size of memory of a computer. They applied the DDDP technique to a system of four reservoirs and showed that the optimal operation strategy can be successfully obtained. But again as the size of the system increases further, even DDDP is not able to manage the problem. To overcome this problem, Nopmongcol and Askew (1976) presented a method called Multilevel

Incremental Dynamic Programming (MIDP). This technique is very much similar to the successive approximation method in which a higher dimensional problem is broken in several subproblems of smaller dimensions which are then solved to obtain the solution. However,this technique has one disadvantage: many times it leads to local optimum instead of a global one. Briefly, the MIDP theory is as follows.

In MIDP, the original system is decomposed into two sub-systems: one active and one inactive. At the first level of computations, only one component is chosen as active and rest are inactive. For example,in a system of ten reservoirs. One reservoir will be active at a time in the first level of computations and rest nine will form the inactive system. The objective function is optimized by adjusting the attributes of only the active subsystem while satisfying all the constraints. When no further better adjustment in the attributes of this active component is possible, this component is put into inactive subsystem and one component from inactive subsystem is made active. The computations are performed in similar manner as given above by adjusting attributes of this component. The process is repeated again until all the components have been activated at least once. Now the second level of computations is taken up. In this level also, the procedure is same except that now two components are made active instead of one. This two-at-a time optimization is over when all the possible pairs of components have become active once. The computations continue for higher levels until no further improvement in the performance is possible. It can be seen that the lower levels act as screens which eliminate suboptimal solutions thus leaving gradually limited region for higher levels. Further, for a high order problem, it is not generally necessary to go to a quite high level of computation. One criterion is to stop the computations when the current level does not give significantly better results than the previous level. The authors successfully applied the MIDP technique to a hypothetical system to show its computational efficiency.

Becker and Yeh(1974) developed an algorithm for optimization of real-time operation of a multiple reservoir system. It was assumed that various uses and purposes are quantifiable via a set of constraints which take care of them, e.g.,storage allocation for flood control purposes, for hydroelectric generation and for recreation uses etc. They defined the best state in each period as the set of reservoir storages which possess the largest total stored potential energy relative to the installed power houses. One attractive feature of the work was that they used a LP-DP formulation in their study. The LP formulation was used to determine the optimal reservoir releases and storage states for each period of the total time interval of interest. It had an objective function which minimized the stored potential energy losses. The stored energy in the reservoirs was expressed in terms of the states of reservoirs and the corresponding energy rate functions. The LP algorithm was used to generate solutions for different values of the on-peak energy generation constraint and corresponding alternative paths from starting vector to ending vector space. Then a DP formulation was used to select from these alternatives an optimal path from any period i to each of the incremental energy levels of period (i+1). This can be done by choosing the alternative which maximizes the sum of components of the end-of-period storage vector. The cumulative energy was chosen as the state variable for the DP algorithm while the feasible energy constraint for a period formed the decision variable. The final result was a policy which maximized total on-peak energy generation over and above contract levels and produced a satisfactory starting state for the next set of periods to be considered. The algorithm was illustrated by an application to the Central Valley Project of California.

Hirsch et al (1977) evaluated synergistic gains from integrated operation of a water resources system. Synergistic gain is defined as gain in benefits due to joint operation of a system of reservoirs, in excess of the benefits from optimal individual operation. These gains can be divided into two components: deterministic and stochastic. These gains are captured by employing operating policy which attempts to minimize the spilling of water from any reservoir when there is a storage capacity available elsewhere in the system. Operating rules like space rule etc. proposed by Maass et al (1962) can be used to accomplish it. It was shown that synergistic gains were significant when daily decisions were taken instead of monthly since short time step allows the rule to react to the changes in relative sizes of inflows.

3.6.1 Stochastic dynamic programming techniques

A better insight of the reservoir problem is obtained if one considers the stochasticity of the inflows in the problem formulation. The DP algorithm can be adapted in two ways to incorporate this stochasticity.

One of them is called Monte Carlo Dynamic Programming (MCDP) suggested by Young (1967) which has been described earlier. An alternate procedure is one which was proposed by Butcher (1971) where stochastic behaviour of inflows is considered explicitly.

Butcher considered one month as the basic time unit for inflows. Any other smaller time unit may also be considered but this will increase the dimensionality of the problem. The sequence of monthly streamflows was regarded to be connected by twelve sets of different transitional probabilities to form a non-stationary Markov Chain. It was also assumed that the system is ergodic, i.e., the steady state of the reservoirs system is independent of the starting state. The storage contents of the reservoir s and the flow q were assumed to be state variables and the release or draft d as the decision variable at any stage. The optimal policy formed by these drafts is assumed to be converged when the values of d's repeat for all values of stage variable i, as i becomes larger. This methods of determining the optimal operating policy is called a 'Policy-iteration routine'. The ergodicity of the process is essential for use of policy-iteration routine. The objective function used in his study was:

$$f_i(s_i,q_{i+1}) = \underset{d}{\mathrm{Max}} \, [\Sigma \, P\,(q_i\,/q_{i+1})[R(d)+\frac{1}{1+r}\, f_{i-1}(s_{i-1},q_i)\;] \qquad ...(117)$$

where

$f_i(s_i,q_{i+1})$= expected return from the optimal operation of a system which has i time periods to the end of the planning period,

s_i = the volume in storage at the start of the i^{th} time period,

q_i = the inflow to the reservoir during the i^{th} time period,

d= release from the reservoir during current time period,

R(d)=return obtained by releasing a quantity of water d in the i^{th} time period,

$P(q_i\, q_{i+1})$ = transition probability connecting the flow in the i^{th} time period with the flow in $(i+1)^{th}$ time period, and

r = the interest rate.

It may be mentioned that for a stationary Markov Chain the optimal policy will change with the changes of interest rate though the changes may be small. The above procedure was successfully applied to a reservoir system by Butcher (1971)

3.7 Multiobjective Optimization

Water resources planning problems deal with the allocation of water among several uses. Traditionally, planners have used a single objective for explicit economic analysis which tries to maximize net benefits. However, the development of system analysis techniques has also made multiobjective analysis techniques popular. Before proceeding further, terms like multipurpose, multiobjective and multiattribute etc. are defined.

Multiple attribute decision problems deal with choosing among a set of alternatives which are described in terms of their attributes. Choosing a particular alternative from a group by such attributes as initial cost, size, capabilities, OMR costs and economy, is a typical example of multiple attribute decision problem. Most of the techniques dealing with these type of problems require the decision maker's preference among values of a given attribute and also across the attributes.

Multiple objective decision models require preference information about the decision maker's objectives and relationship between objectives and attributes. Preferences among attributes are derived from the preferences among objectives and the functions relatingattributes to objectives. In these models, an alternative can be described either in terms of its attributes or in terms of the extent to which it achieves the objectives of the decision maker. For example, the decision maker's relevant objectives may be safety against flood,recreation facilities ,dependable water supply etc. A multiobjective model would require priorities on values of cost versus capacity and also the linkages which relate the extent to which initial cost, size etc. contribute to dependability due to these relations . The multiobjective models are more complex than the multiattribute models. These two are, however, same when there is one-to-one relationship between attributes and objectives.

The term multipurpose means a project or implement for several purposes like irrigation flood control and power etc. These can be designed to serve a single objective, e.g.,increasing the income of a region. Similarly a navigation project which is single purpose project may be aimed towards objectives of increasing regional income and national economic growth. Thus terms multiobjective and multipurpose are not synonyms ; several purposes can be aimed towards one objective and one purpose towards several objectives. But it often happens that the multiple purposes for which a project is planned are not complementary in nature and the linkages or functional relationships among the purposes the objectives are not clearly known. In such cases, the multiple purposes themselves may be treated as multiple objectives.

Because of involvement of a vector in the objective function, the technique of multiobjective optimization is also called'vector optimization'. The problem can be stated as

$$\text{Maximize } F(x) = \text{Maximize } [\, f_1(X) \; f_2(X).....f_n(x) \,] \qquad ...(116)$$

Subject to $g_j(X) \leq 0$, $j=1, 2.....m$

where

$F(X)$ = vector valued objective function,

X = an N dimensional feasible vector of decision variables,

$f_i(x)$= n objectives functions which are components of $F(X)$, and

$g_j(X)$ = set of constraints.

A decision X^* is said to be noninferior solution if there does not exist another $\bar{X}$ such that

$$f_i(\bar{X}) \geq f_i(X^*), \quad i=1.....n \qquad ...(119)$$

Strict inequality should hold for at least one i value. The case of two objectives is represented graphically in figure 13.

The curve AE represents the boundary of the set of feasible alternatives and is called net benefit transformation curve or TC. Points which lie between TC and origin are called feasible points and those outside are called infeasible points. A point M between origin and TC is worse than at least one point on TC. For example, point M is a feasible point but is worse than points B and C. The value of $f_2(X)$ can be still increased moving from M to C keeping $f_1(X)$

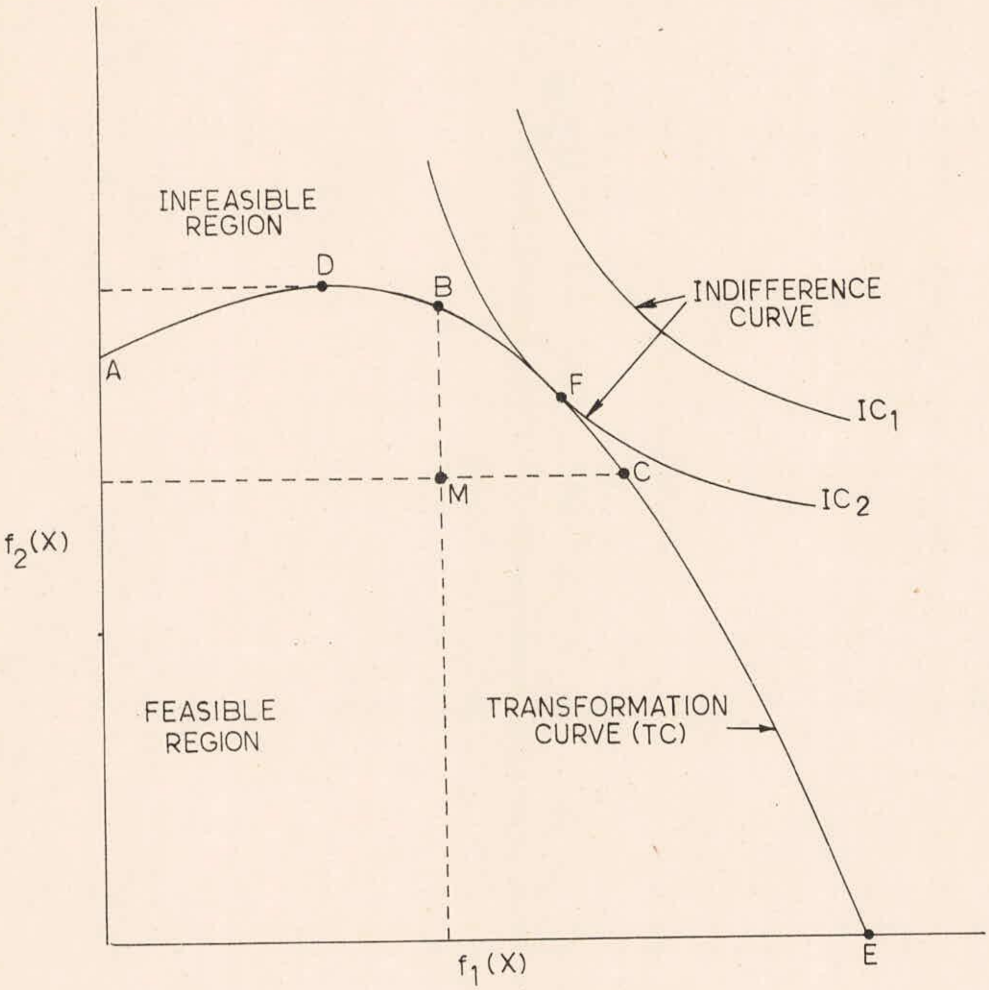

FIGURE -13 GRAPHICAL PRESENTATION OF MULTIOBJECTIVE ANALYSIS FOR TWO OBJECTIVES same. Similarly $f_2(X)$ can also be increased by moving from M to B without affecting $f_1(X)$. Curve AE is called non-inferior set. Each point on the curve DE represents one non-inferior alternative to decision maker. The preferences of society are represented by indifference curves. The optimal point will be at the tangency of the highest attainable indifference curve (IC_2 in present case) and TC.

It is clear from above discussions that to find out the optimal solution, first net benefit transformation curves and social indifference curves need to be derived. However, it has been pointed out in many studies that indifference curves are very difficult to draw for practical applications.

Non -inferior solution set can be generated by two techniques. In first approach called Parametric approach, vector objective function formulation is replaced by

$$\underset{X}{\text{Max}} \sum_{i=1}^{n} w_i \, f_i \, (X) \qquad (120)$$

subject to

$$g_j \, (X) \leqslant 0, \quad j=1,2,\ldots\ldots m \qquad \ldots(121)$$

where w is vector of weighting coefficients

$$w = [w_1 \; w_2 \; \ldots\ldots w_n \;]$$

with

$$w_i \geqslant 0 \qquad \text{and} \qquad \sum_{i=1}^{n} w_i = 1 \qquad \ldots(122)$$

The problem is solved parametrically for different combinations of w_i

In another approach, called constraint approach, all except one of the objective functions are replaced by constraints as follows:

$$\text{Max} \quad f_i \, (X) \qquad \ldots(123)$$

Subject to $g_j \, (X) \leqslant 0$, j=1,2.....m ...(124)

$$f_j \, (X) \geqslant \varepsilon_j \qquad j=1,2.,\ldots\ldots m \text{ and } j \neq i$$

ε_j, = 1,2.....n,j $\neq$ i are the minimum tolerable levels of (n-1) objectives. The ε_j values can be varied and their impact can be studied. Similarly, the ith objective $f_i(X)$ can be replaced by jth objective and the solution procedure can be repeated. The method can map the entire non inferior solution set even when it is nonconvex.

The methods described above are suitable only when there are two or three objectives. Several other methods have been developed to solve more complex problems. Haimes and Hall (1974) developed a very powerful technique called Surrogate Worth Trade-off Method. Another technique which is gradually becoming popular is Goal programming. These two techniques are being discussed here.

3.7.1 Surrogate Worth Trade-off Method

It was pointed out by Major (1969) that the line which passes through the point of tangency of non inferior set and the social indifference curve is a surrogate for preference with respect to the two objectives. The negative of the slope of this line represents marginal trade-off among these objectives and is called weight. These weights represent marginal importance of the society for the objectives (which also depends upon the degree of fulfilment of objectives at the point from which the marginal departures are measured.)

Referring to the constraint method, the Lagrange multipliers related with (n-1) objectives as constraints may be zero or non-zero. If Lagrange multiplier is non zero for a constraints, that particular constraint does limit the optimum. Non zero Lagrange multipliers correspond to noninferior set of solutions while zero Lagrange multipliers correspond to inferior set of solutions. Moreover, the set of non-zero Lagrange multipliers represents the set of trade-off ratios between the principal objective and each of constraining objectives.

The trade off function between ith and jth objective function is denoted by $T_{ij}(X)$ and is defined as:

$$T_{ij}(X) = df_i(X)/df_j(X) \qquad \text{...(125)}$$

where $$df_i(X) = \sum_{k=1}^{n} \frac{\partial f_i(X)}{\partial x_k} dx_k \qquad \text{...(126)}$$

Thus

$$T_{ij}(X) = \frac{(\nabla X\, f_i(X), dX)}{(\nabla X\, f_j(X), dX)} \qquad \text{...(127)}$$

The functions $T_{ij}(X)$ have the property that

$$T_{ij}(X) = 1, \text{ if } i = j \qquad \text{...(128)}$$

$$T_{ij}(X) = 1/T_{ji}(X) \text{ for all } i,j$$

Rewriting constraint method formulation as:

$$\text{Max} \quad f_i(X)$$

Subject to

$$f_j(X) \leqslant \varepsilon_j \quad j=2,3....n \qquad \text{...(129)}$$

$$g_k(X) \leqslant 0 \quad k=1,2.....m$$

where

$$\varepsilon_j = \bar{f}_j(X) + \bar{\varepsilon}_j \quad = 2,3.....n$$

$$\bar{\varepsilon}_j \geqslant 0 \qquad j = 2,3,......m \qquad \text{...(130)}$$

$$\bar{f}_j(X) = \min \quad f_j(X)$$

Now the generalized Lagrangian L to the system can be formulated as

$$L = f_i(X) + \sum_{k=1}^{m} \mu_k g_k(X) + \sum_{j=2}^{n} \lambda_{ij}(f_j(X) - \varepsilon_j) \qquad \text{...(131)}$$

where

μ_k, k=1,.....m and λ_{ij}, j=2,3.....n are generalised Lagrange multipliers. The notation λ_{ij} denotes that λ is the Lagrange multiplier associated (in the ε constraint vector optimization problem) with the jth constraint where the objective function is $f_i(X)$. Denoting by X the set of all x_i i=1,2....n that satisfy the Kuhn-Tucker condition in eqn.(131) and by Ω the set of all Lagrange multipliers that satisfy the Kuhn-Tucker conditions, the Kuhn-Tucker conditions for stationary values of X, μ_k and λ_{ij} (k=1....m,j=2.....n) which are of interest are:

$$\lambda_{ij} \; (f_j(X) - \varepsilon_j) = 0 \quad j=2,3.....n \qquad ...(132)$$

$$\lambda_{ij} \geqslant 0 , \quad j=2,3.....n$$

This holds good only if $\lambda_{ij} = 0$ or $f_i (X)-\varepsilon_j = 0$ or both. The case $\lambda_{ij}=0$ represents that constraint is not binding. The case $\lambda_{ij} \neq 0$ represents a bind or active constraint for which it can be shown that

$$\lambda_{ij}[A (\varepsilon_j)] = - \partial f_i(X)/\partial f_j(X) \qquad ...(133)$$

$$i \neq j, i,j=1,2.....n$$

where $\lambda_{ij}[A (\varepsilon_j)]$ Lagrange multiplier for active constraint associated with specific value of ε_j.

The above relationship is valid only when the jth constraint is active. It can be shown that active and nonactive Lagrange multipliers have direct correspondence with noninferior and inferior solution sets respectively.

As objectives i and j may be non commensurable, it may not be possible to compare the worth of $\Delta f_i(X)$, with the true worth of $\Delta f_j(X)$. A surrogate worth function w_{ij}, i,j=1,2....n, $i \neq j$, can be defined as an ordinal monotonic function satisfying following

$W_{ij} > 0$ when λ_{ij} marginal units of $f_i(X)$ are preferred over one marginal unit of $f_j(X)$ given the level of achievement of all objectives.

$W_{ij} = 0$ when λ_{ij} marginal units of $f_i(X)$ are equivalent to one marginal unit of $f_j(X)$ given the level of achievement of all objectives.

$w_{ij} < 0$ when λ_{ij} marginal units of $f_i(X)$ are not preferred over one marginal unit of $f_j(X)$ given the level of achievement of all objectives.

The optimal solution is reached when $w_{ij}=0$ for all values of i and j, $i \neq j$.

One striking feature of SWT method is that the surrogate functions which relate the decision maker's preferences to the non-inferior solutions through the trade-off functions are constructed in the functional space and then transformed into the decision space. Here non-commensurable objective functions can be handled quantitatively. The method is computationally feasible and tractable.

Cohon and Marks(1975) reviewed and evaluated various multiobjective programming techniques. Following three criteria were chosen by them for comparison:

(a) The technique must be computationally efficient and relatively feasible.

(b) It must foster the explicit quantification of trade-offs among objectives.

(c) It must provide sufficient information so that an informed decision can be made.

Among the various techniques considered by them were Weighting and Constraints method, Goal programming, Electre-method, Surrogate Worth Trade-off method, and Step method. They concluded that when there are fewer than four objectives, a generating technique such as Weighting method or Constraint method should be used inorder to capture the essence of the multiobjective problem. When there are four or more objectives a technique which restricts the size of the feasible region such as Surrogate Worth Trade-off method should be used.

3.7.2 Goal programming

Based upon previous discussion, it can be concluded that most real life problems have more than one objective. Further,many times these objectives may be conflicting in nature. In the optimization techniques generally used to solve the problem, one objective is sometimes treated as primary and others are treated as constraints. The optimum solution of the problem must satisfy all constraints. Further, all constraints are considered as equally important. If any one of the constraints is not satisfied at a particular point, that point is called infeasible. It might be that the particular constraint has been violated by a very small margin and the objective function would have been significantly higher otherwise at this point then the value found. There are very good chances that this fact would be missed unless, say in case of linear programming, a sensitivity analysis is performed carefully. The important question to be pondered over is whether an important alternative should be dropped just because it has been declared infeasible. For example, a reservoir has to be constructed which is required to always supply 100 units of water to a city. Suppose we have one alternative which is otherwise very much suitable but during one month, it can only supply 99 units of water. Solving this problem with linear programing would label this alternative as infeasible because lower bound is violated once. However, it will be agreed that to reject this choice would be an unwise decision. Goal Programming is very much useful in this type of cases.

Goal Programming is an extension of linear programming. Problems involving one or more goals can be solved using it. In many reservoir operation problems, the goals are conflicting, like in case of multipurpose reservoir operated for irrigation and flood control. The goals may also be incommensurable which means that they can not be measured in same units. First, it is required to arrange these goals in hierarchy, i.e.,in order of their preference by putting top priority goal first, and so on. It would be desirable to satisfy the top priority goal first and then to take up next priority goal. Generally, it is very difficult to rank the goals in cardinal order. The Goal progoramming allowes for an ordinal solution to the problem.

The technique of Goal progoramming has been very much useful in modern management too. It has been variously pointed out that nowadays the aim of management is not to optimize but to satisfy. An optimizer goes for the best possible outcome. While a satisfier tries to attain satisfactory level of multiple objectives. Goal programming approach is based upon these concepts.

The deviations from attainment of each goal are represented by two types of variables: one represents positive deviations and other represents negative deviations. In Goal programming formulation the idea is to attempt to minimize these deviations.

In the Goal programming notation, any variable, which may be positive, negative or zero, can be represented as the difference of two positive variables, i.e.,

$$x = x^+ - x^- \qquad \text{...(134)}$$

where

$$x^+ = x \quad \text{if } x \geq 0$$

$$x^+ = 0 \quad \text{if } x \leq 0$$

$$x^- = -x \quad \text{if } x \leq 0$$

$$x^- = 0 \quad \text{if } x \geq 0$$

It is also evident from the above that

$$x^+ . x^- = 0 \qquad \text{...(136)}$$

Let us consider that a hypothetical reservoir is to be constructed at a site and the goals representing cost of construction, minimum flow requirement and flood control benefit are expressed by

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \leq b_1 \qquad \text{...(137a)}$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \geq b_2 \qquad \text{...(137b)}$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 \geq b_3 \qquad \text{...(137c)}$$

Let us introduce three new variables y_1, y_2 and y_3 such that

$$y_1 = a_{11} x_1 + a_{12} x_2 + a_{13} x_3 - b_1 \qquad \text{...(138a)}$$

$$y_2 = a_{21} x_1 + a_{22} x_2 + a_{23} x_3 - b_2 \qquad \text{...(138b)}$$

$$y_3 = a_{31} x_1 + a_{32} x_2 + a_{33} x_3 - b_3 \qquad ...(138c)$$

and

$$y_1 = y_1^+ - y_1^- \qquad y_1^+, y_1^- \geq 0 \qquad ...(139a)$$

$$y_2 = y_2^+ - y_2^- \qquad y_2^+, y_2^- \geq 0 \qquad ...(139b)$$

$$y_3 = y_3^+ - y_3^- \qquad y_3^+, y_3^- \geq 0 \qquad ...(139c)$$

Now let us assume that the penalty for exceeding first goal is C_1, for shortage in second goal is C_2 and for shortage in third goal is C_3. Hence objective function and constraints can be written in the following form:

$$\text{Min } Z = C_1 y_1^+ + C_2 y_2^- + C_3 y_3^- \qquad ...(140)$$

Subject to

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 - (y_1^+ - y_1^-) = b_1 \qquad ...(141a)$$

$$a_{12} x_1 + a_{22} x_2 + a_{23} x_3 - (y_2^+ - y_2^-) = b_2 \qquad ...(141b)$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 - (y_3^+ - y_3^-) = b_3 \qquad ...(141c)$$

$$x_i \geq 0, \quad y_i^+ \geq 0, \quad y_i^- \geq 0$$

$$i = 1, 2, 3$$

This problem can be easily solved by Simplex method used for solving linear programming problems.

As pointed out earlier, it is easy to arrange the goals in ordinal way rather than cardinal way. It may be pointed out that in cardinal ranking,a goal with priority factor 3 is three times important than goal with priority factor 1, while in ordinal ranking it only means that first goal is mere important than second without implying any relative degree of priority. In such problems, the deviations about the goals are ranked according to pre-emptive priority factors, P_i (i=1,2.....n). These factors have the relationship that $P_1 >>> P_2 >> P_3 .. >>> P_n$. Thus lower order goals are considered only when higher order goals have been satisfied.

In many situations, two objectives or goals may have same priority. Under these circumstances, weightage factors are assigned among the same priority goals. These weights can be estimated using benefit-cost ratio of each objective. Let the priority of minimizing y_1^+ and y_2^- in the above example be C_1. Then the objective function will look like

$$\text{Min } Z = C_1 y_1^+ + C_1 y_2^- + C_3 y_3^- \qquad ...(142)$$

Now we can assign different weights to these two objeetives(which have same priority). The new objective function would be

$$\text{Min } Z = W_1 C_1 y_1^+ + W_2 C_2 y_2^- + C_3 y_3^- \qquad ...(143)$$

where W_1 and W_2 are weights.

Recently Can and Houck(1984) applied goal programming for realtime operation of Green River Basin System which consists of four multipurpose reservoirs. The effectiveness of this technique was compared with a linear programming model. They chose a piecewise linear convex penalty function. The coefficients of penalty were different for different segments of the function with smaller penalty for deviations in the vicinity of the target and higher values in the extreme region. The objective of linear programming formulation was to minimize the sum of penalties. Two dimensional information is required in linear programming model-the boundaries of different segments and the penalty coefficients for each of them. In contrast to this, in goal programming model, one dimensional information is needed only. The user simply specifies the preferred goal. It was observed that the operation policy given by goal programming model resulted in better operating decision. The use of goal program is more simple than the linear program because it is less data intensive. As the estimation of penalty coefficient is very different and further, it may vary from person to person, it introduces subjectivity in problem formulation. This deficiency is not present in goal programming formulation.

4.0 CONCLUSIONS

Linear Programming has been a widely used and useful tool for solving problems of water resources systems. Its application is growing & better algorithms are being developed and computer programs are available for more and investigators . LP can easily tackle problems with high dimensionality and can give global optimal. One of the most explored framework of application of LP has been in the form of Linear Decision Rule. Over the past years, LP is being increasingly used in stochastic framework.

Dynamic Programming is another technique which has been extensively used for problems of our interest. Interestingly, it has also been used in conjunction with LP in some cases. DP does not have any restriction on the nature of objective function and constraint except the separability condition for objective function. However, there are mainly two factors which limit its application. Due to its inherent nature, generalized algorithms for DP are not available. Thus one has to develope and test his own program for a particular situation. Further, dimensionality becomes a tedious problem for bigger systems. Nowadays, efforts are directed towards multiobjective optimization and development of routines for real-time operation problems.

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