DP-7

# POLYNOMIAL REGRESSION

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ABSTRACT

For any non linear function $Y = f(X)$ regression may be obtained by fitting a polynomial. The general form of the polynomial regression is as given under:

$$Y = a_0 + a_1X + a_2X^2 + \ldots\ldots\ldots\ldots\ a_mX^m + \varepsilon$$

where:

$Y$ is the dependent variable and $a_0$, $a_1$ ..... $a_m$ are the regression coefficients. The documentation of the computer programme for polynomial regression includes the listing of the source file, data file and output file with test data and example calculations. The details of various statistics given in the programme output have also been given in the documentation.

## 1.0 INTRODUCTION

Regression represents a mathematical equation expressing one random variable as being correlatively related to another random variable or several random variables. The regression equation may be any function that can be fitted to a set of observed variables. If the variables are linearly related then the regression is called linear regression. In non-linear regression the variables are non-linearly related.

For any non-linear function, $Y = f(X)$ regression may be obtained by fitting a polynomial. The general form of the polynomial regression is as given under:

$$Y = a_0 + a_1X + a_2X^2 \dots\dots\dots\dots + a_mX^m + \varepsilon \qquad \dots(1)$$

The coefficients $a_0, a_1, \dots\dots\dots, a_m$ are the regression coefficients and are determined by the least squares method of parameter estimation. $\varepsilon$ is the error term. The power order M is chosen so as to minimise the sum of the squares of deviations from the line. The power M should be much lower than the sample size, N, in order to have a sufficient number of degrees of freedom (N - M - 1) and to have a reliable estimate of the standard deviation. Generally, the value of M is between 2 and 4 as it is very difficult to explain higher degree of M.

It is preferable to plot the data on a single graph first to have a preliminary idea about the value of M or the degree of polynomial to be fitted to the data and also to eliminate any inconsistent points.

The polynomial regression analysis is generally used for trend analysis in hydrologic data. The programme for polynomial regression analysis described in this documentation has been taken from IBM Scientific Subroutine Package and implemented/tested on VAX-11/780 computer system of National Institute of Hydrology, Roorkee.

## 2.0 PURPOSE OF THE PROGRAMME

The programme calls subroutines to perform the regression analysis. The programme prints the regression coefficients and analysis of variance tables for polynomials of successively increasing degrees. The programme also optionally prints the table of residuals and a plot of observed Y values and Y estimates versus base variable X. If there is no reduction in the residual sum of squares between two successive degrees of polynomials the programme terminates the problem, otherwise it continues till the analysis for the highest degree of polynomial specified is completed.

## 3.0 SPECIFIC METHOD

Various statistical parameters given in the output are computed by the following equations:

a) Regression coefficients for successive degrees of polynomials:

Regression coefficients are the values of $a_1$, $a_2$, $a_3$ etc. in the equation given below:

$$Y = a_0 + a_1X + a_2X^2 + \ldots\ldots\ldots\ldots + a_mX^m \qquad \ldots(2)$$

For first degree polynomial there will be only one regression coefficient. The value of $a_0$ is the intercept.

b) Analysis of variance for successive degrees of polynomials:

Analysis of variance includes computation of variance explained due to regression and due to deviation about the regression line. The F-value is also computed.

i) Variance due to regression is given by the following expression:

$$\text{Variance due to regression} = \sum_{i=1}^{N} (Y_i' - \bar{Y})^2 \qquad \ldots(3)$$

where:

$Y_i'$ : Estimated $i^{th}$ value of Y

$\bar{Y}$ : The mean of Y values

ii) Variance due to deviation about regression is given by:

$$\text{Variance due to deviation about regression} = \sum_{i=1}^{N} (Y_i - Y'_i)^2 \qquad \ldots(4)$$

where:

$Y_i$ : Observed $i^{th}$ value of Y

$Y'_i$ : Estimated $i^{th}$ value of Y

c) F value :

F value is the ratio of mean squares due to regression and mean squares about the regression line.

$$\text{F value} = \frac{\text{Mean squares due to regression}}{\text{Mean squares about regression}} \qquad \ldots(5)$$

## 4.0 COMPUTER PROGRAMME

The main features of computer programme and different subroutines are discussed as follows:

### 4.1 Programme Subroutines

The computer programme for polynomial regression consists of the main routine named PREG and five other subroutines named GDATA for data matrix generation for polynomial regression, ORDER for rearrangement of intercorrelations, MINV for matrix inversion, MULTR for multiple linear regression and PLOT for plotting. These subroutines are described in following paragraphs:

A. Subroutine GDATA (N, M, X, XBAR, STD, D, SUMSQ)

This subroutine generates independent variables upto the $M^{th}$ power ( the highest degree polynomial specified) and calculates means, standard deviations, sums of cross products of deviations from means and product moment correlation coefficients. The calling arguments are:

N : Number of observations

M : The highest degree polynomial to be fitted

X : Input matrix (N x M+1). When the subroutine is called, data for the independent variable are stored in the first column of the matrix, and data for the dependent variable are stored in the last column of the matrix. Upon returning to the calling routine, generated powers of the independent variable are stored in the columns 2 through M

XBAR : Output vector of length M + 1 containing means of independent and dependent variables

STD: Output vector of length M+1 containing standard deviations of independent and dependent variables

D : Output matrix containing correlation coefficients

SUMSQ: Output vector of length (M+1) containing sums of products of deviations from means of independent and dependent variable

B. Subroutine ORDER ( M,R, NDEP, K, ISAVE, RX, RY)

The purpose of this subroutine is to construct from larger matrix of correlation coefficients, a subset of matrix of intercorrelations among independent variables and a vector of intercorrelations of independent variables with dependent variable. The calling arguments are:

M : Number of variables and order of matrix R

R : Input matrix containing correlation coefficients

NDEP: The subscript number of the dependent variable

K : Number of independent variables to be included in the forthcoming regression, K must be greater than or equal to 1.

ISAVE: Input vector of length K + 1 containing, in ascending order, the subscript numbers of K independent variables to be included in the forthcoming regression upon returning to the calling routine. This vector contains, in addition, the subscript number of the dependent variable in K+1 position

RX: Output matrix ( K x K ) containing intercorrelations among independent variables to be used in forthcoming regression

RY: Output vector of length K containing intercorrelations of independent variables with dependent variable

C. Subroutine MINV ( A, N, D, L, M)

The subroutine is used for matrix inversion. The calling arguments are:

A : Input matrix destroyed in computation and replaced by resultant inverse

N : Order of matrix A

D : Resultant determinant

L : Work vector of length N

M : Work vector of length N

D. Subroutine MULTR (N, K, XBAR, STD, D, RX, RY, ISAVE, B,SB, T, ANS)

This performs a multiple linear regression analysis for a dependent variable and a set of independent variables.

The calling arguments are :

N : Number of observations

K : Number of independent variables in the regression

XBAR : Input vector of length M containing means of all variables. M is the number of variables in the observations

STD: Input vector of length M containing standard deviations of all variables

D : Input vector of length M containing the diagonal of the matrix of sums of cross products of deviations from means for all variables

RX: Input matrix ( K x K) containing the inverse of intercorrelations among independent variables

RY: Input vector of length K containing intercorrelations of independent variables with dependent variable

ISAVE: Input vector of length (K+1) containing subscripts of dependent variables in ascending order. The subscript of the dependent variable is stored in the last K+1 position

B : Output vector of length K containing regression coefficients

SB : Output vector of length K containing standard deviations of regression coefficients

T : Output vector of length K containing t values

ANS: Output vector of length 10 containing the following information

ANS(1) : Intercept

ANS(2) : Multiple correlation coefficient

ANS(3) : Standard error of estimate

ANS(4) : Sum of squares attributable to regression (SSAR)

ANS(5) : Degree of freedom associated with SSAR

ANS(6) : Mean squares of SSAR

ANS(7) : Sum of squares of deviations from regression (SSDR)

ANS(8) : Degrees of freedom associated with (SSDR)

ANS(9) : Mean squares of SSDR

ANS(10): F value

E. Subroutine PLOT (NO, A, N, M, NL, NS )

The purpose of this subroutine is to plot Y values and Y estimates versus base variable X. The calling arguments are :

NO : Chart number

A : Matrix of data to be plotted. First column represents base variable and successive columns are the cross variables ( maximum 9)

N : Number of rows in A

M : Number of columns in matrix A (equal to the total number of variables ) maximum is 10

NL : Number of lines in the plot. If 0 is specified 50 lines are used

NS : Code for sorting the base variable data in ascending order

0 : Sorting is not necessary

1 = Sorting is necessary

The listing of the source programme has been given in Appendix I.

### 4.2 Programme Modifications

The programme capacity can be increased or decreased by making changes in dimension statements. The following are the general rules for the programme modifications:

a. The dimension of array X must be greater than or equal to the product N (M + 1), where N is the number of observations and M is the highest degree polynomial to be fitted.

b. The dimensions of array DI must be greater than or equal to the product of M x M.

c. The dimension of array D must be greater than or equal to (M+2) (M+1)/2.

d. The dimensions of arrays B, E, SB and T must be greater than or equal to the highest degree polynomial to be fitted, M.

e. The dimension of arrays XBAR, STD, COE, SUMSQ and ISAVE must be greater than or equal to (M+1).

f. The dimension of array P must be greater than or equal to 3 x N.

## 5.0 INPUT SPECIFICATIONS, OUTPUT DESCRIPTION AND RESCTIRCTIONSON USE

### 5.1 Input Specifications

Input data file contains control and data lines/cards.

#### 5.1.1 Control cards

In first line, title of the problem is given in A format.

Second line

| Column | Contents |
|---|---|
| 1-5 | Number of observations |
| 6-7 | Highest degree polynomial to be fitted |
| 8 | Option code for plotting Y values and Y estimates versus base variable X |
| | 0 if it is not required |
| | 1 if it is required |

#### 5.1.2 Data cards

Input data are read into the computer one observation at a time i.e. each pair or X and Y values in free format.

### 5.2 Output Description

The output for the programme for polynomial regression includes:

a. Regression coefficients for successive degree polynomials

b. Analysis of variance for successive degree polynomials

c. Table of residuals for the final degree polynomial

d. Plot of Y values and Y estimates versus base variable X (Optional). Observed Y values are denoted by 1 and estimated values by 2.

5.3 Restrictions on Use

All the pairs should be complete i.e. there should not be any missing data.

## 6.0 TEST DATA

The programme for polynomial regression has been run on 93 years (1887-1979) annual peak stage data of river Narmada at BROACH. This has been done in order to see the trend in the data. The stage data is given below:

| Sl No. | Year | Stage (m) | Sl No. | Year | Stage (m) | Sl No. | Year | Stage (m) |
|---|---|---|---|---|---|---|---|---|
| 1. | 1887 | 26.50 | 35. | 1921 | 24.50 | 69. | 1955 | 28.50 |
| 2. | 1888 | 21.00 | 36. | 1922 | 24.50 | 70. | 1956 | 27.00 |
| 3. | 1889 | 23.50 | 37. | 1923 | 30.00 | 71. | 1957 | 27.50 |
| 4. | 1890 | 25.00 | 38. | 1924 | 27.00 | 72. | 1958 | 25.50 |
| 5. | 1891 | 22.50 | 39. | 1925 | 22.00 | 73. | 1959 | 33.00 |
| 6. | 1892 | 27.50 | 40. | 1926 | 29.50 | 74. | 1960 | 25.50 |
| 7. | 1893 | 24.50 | 41. | 1927 | 28.00 | 75. | 1961 | 25.20 |
| 8. | 1894 | 30.00 | 42. | 1928 | 27.50 | 76. | 1962 | 32.30 |
| 9. | 1895 | 21.50 | 43. | 1929 | 25.00 | 77. | 1963 | 23.00 |
| 10. | 1896 | 30.00 | 44. | 1930 | 30.50 | 78. | 1964 | 25.25 |
| 11. | 1897 | 27.00 | 45. | 1931 | 29.50 | 79. | 1965 | 23.00 |
| 12. | 1898 | 30.00 | 46. | 1932 | 29.00 | 80. | 1966 | 20.60 |
| 13. | 1899 | 18.25 | 47. | 1933 | 34.00 | 81. | 1967 | 23.75 |
| 14. | 1900 | 28.00 | 48. | 1934 | 29.50 | 82. | 1968 | 38.50 |
| 15. | 1901 | 26.50 | 49. | 1935 | 23.50 | 83. | 1969 | 29.60 |
| 16. | 1902 | 20.50 | 50. | 1936 | 26.50 | 84. | 1970 | 41.50 |
| 17. | 1903 | 23.50 | 51. | 1937 | 30.50 | 85. | 1971 | 26.00 |
| 18. | 1904 | 21.50 | 52. | 1938 | 30.50 | 86. | 1972 | 32.00 |
| 19. | 1905 | 30.70 | 53. | 1939 | 34.00 | 87. | 1973 | 36.50 |
| 20. | 1906 | 32.50 | 54. | 1940 | 26.50 | 88. | 1974 | 29.98 |
| 21. | 1907 | 25.00 | 55. | 1941 | 29.00 | 89. | 1975 | 28.50 |
| 22. | 1908 | 24.50 | 56. | 1942 | 34.50 | 90. | 1976 | 26.50 |
| 23. | 1909 | 17.00 | 57. | 1943 | 25.50 | 91. | 1977 | 26.99 |
| 24. | 1910 | 25.50 | 58. | 1944 | 35.00 | 92. | 1978 | 29.54 |
| 25. | 1911 | 20.50 | 59. | 1945 | 33.00 | 93. | 1979 | 31.49 |
| 26. | 1912 | 28.00 | 60. | 1946 | 27.50 | | | |
| 27. | 1913 | 33.00 | 61. | 1947 | 31.50 | | | |
| 28. | 1914 | 21.50 | 62. | 1948 | 29.50 | | | |
| 29. | 1915 | 24.50 | 63. | 1949 | 29.00 | | | |
| 30. | 1916 | 27.50 | 64. | 1950 | 35.00 | | | |
| 31. | 1917 | 27.00 | 65. | 1951 | 23.50 | | | |
| 32. | 1918 | 19.50 | 66. | 1952 | 23.50 | | | |
| 33. | 1919 | 32.50 | 67. | 1953 | 33.00 | | | |
| 34. | 1920 | 22.50 | 68. | 1954 | 29.50 | | | |

## 7.0 EXAMPLE CALCULATIONS

The calculations for first degree polynomial are given below:

Step 1 : For first degree polynomial (straight line) regression coefficients are obtained by method of least squares. The coefficients are given below:

Intercept = 24.62

Regression coefficient $a_1$ = 0.0611

Step 2 : Sum of squares due to regression = $\sum_{i=1}^{93} (Y_i' - \bar{Y}_i)^2$

$$= 250.37$$

Sum of squares due to deviation from regression

$$= \sum_{i=1}^{93} (Y_i' - Y_i)^2 = 1615.39$$

$$F \text{ value} = \frac{\text{Mean squares due to regression}}{\text{Mean squares due to deviation from regression}}$$

$$= \frac{(250.37/2)}{(1615.39/91)} = 14.10$$

## 8.0 APPLICATION, SAMPLE INPUT AND SAMPLE OUTPUT

The programme for polynomial regression has been run on the 93 years peak stage data of river Narmada at Broach to analyse the trend in the data.

### 8.1 Sample input

The listing of the sample input ( data file ) has been given in Appendix II.

In the sample problem total number of observations is 93 and the highest degree of polynomial required is 2. The plot is also required. The value of N, M and N PLOT will be 93, 2, 1 respectively.

In the data lines, the value of X will vary from 1 to 93. The X values and Y values have been punched in free format.

### 8.2 Sample Output

The listing of the sample output has been given in Appendix III.

The programme first fits one degree polynomial to the stages. The relation obtained is

$$Y = 24.623 + 0.0611207\ t$$

The deviation about regression line is 1615.3916. In case of second degree polynomial, the relation obtained is

$$Y = 23.83937 + 0.11062\ t - 0.0005267\ t^2$$

The deviation about regression line is 1604.6779. The improvement in terms of sum of squares for second degree polynomial is 10.71358. So second degree polynomial better fits the data as compared to first degree. The table of residuals and plot of Y values denoted by (1) and Y estimates denoted by (2) versus base variable X has been given in Appendix III.

## 9.0 RECOMMENDATIONS

The programme for polynomial regression can deal upto 100 observations and $10^{th}$ degree polynomial. Therefore if a problem satisfies the above conditions it is not necessary to modify the programme. However, if there are more than 100 observations or if greater than $10^{th}$ degree polynomial is desired, dimension statements in the main programme must be modified to handle the problem.

## REFERENCES

1. Haan, C.T. (1977), 'Statistical Methods in Hydrology', Iowa State University Press, Ames, Iowa.

2. Scientific Subroutine Package, International Business Machines, White Plains, N.Y.

APPENDIX I

## POLYNOMIAL REGRESSION PROGRAMME

```
C       MASTER POLYNOMIAL REGRESSION
        DIMENSION X(1100),DI(100),D(66),B(10),E(10),SB(10),
       1T(10),XBAR(11),STD(11),COE(11),SUMSQ(11),ISAVE(11),
       2ANS(10),P(300),TITLE(80)
        OPEN(UNIT=5,FILE='PREG.DAT',STATUS='OLD')
        OPEN(UNIT=6,FILE='PREG.OUT',STATUS='NEW')
 1      FORMAT(I5,I2,I1)
 2      FORMAT(2F6.0)
 3      FORMAT(1X' POLYNOMIAL REGRESSION.......'/)
 4      FORMAT(/,' NUMBER OF OBSERVASIONS',I6/)
 5      FORMAT(/' POLYNOMIAL REGRESSION OF DEGREE',I3)
 6      FORMAT(' INTERCEPT',F20.4)
 7      FORMAT(/,' REGRESSION COEFFICIENTS'/(6F20.4))
 8      FORMAT(1H0/20X,' ANALYSIS OF VARIENCE FOR',I4,
       1'DEGREE OF POLYNOMIAL')
 9      FORMAT(1H0,1X,'SOURCE OF VARIATION',5X,'DEGREE OF'
       1,3X,'SUM OF',3X,4HMEAN,5X,1HF,4X,'IMPROVEMENT IN
       2TERMS'/26X ,'FREEDOM',4X,'SQUARES',3X,'SQUARE',3X,
       35HVALUE,1X,'OF SUM OF SQUARES')
 10     FORMAT('ODUE TO REGRESSION',7X,I6,F13.2
       1,F10.2,F7.2,F12.2)
 11     FORMAT(' DEVIAION ABOUT REGRESSION',I5,F13.2,F10.2)
 12     FORMAT(8X,'TOTAL',11X,I6,F13.2///)
 13     FORMAT('      NO IMPROVEMENT')
 14     FORMAT(1H0//27X,' TABLE OF RESIDUALS'//
       116 HOBSERVATION NO.,5X,7HX VALUE,7X,7HY VALUE,7X,
       210HY ESTIMATE,7X,8HRESIDUAL/)
 15     FORMAT(3X,I6,F18.5,F14.5,F17.5,F15.5)
 16     FORMAT(80A1)
100     READ(5,16)TITLE
        READ(5,1)N,M,NPLOT
        WRITE(6,16)TITLE
        WRITE(6,3)
        WRITE(6,4)N
C       READ INPUT DATA
        L=N*M
        DO 110I=1,N
        J=L+I
 110    READ(5,*) X(I),X(J)
        CALL GDATA(N,M,X,XBAR,STD,D,SUMSQ)
        MM=M+1
        SUM=0.0
        NT=N-1
        DO 200 I=1,M
        ISAVE(I)=I
        CALL ORDER(MM,D,MM,I,ISAVE,DI,E)
        CALL MINV(DI,I,DET,B,T)
        CALL MULTR(N,I,XBAR,STD,SUMSQ,DI,E,ISAVE,B,SB,T,ANS)
        WRITE(6,5) I
        IF(ANS(7)) 140,130,130
```

```
130   SUMIP=ANS(4)-SUM
      IF(SUMIP) 140,140,150
140   WRITE(6,13)
      GO TO 210
150   WRITE(6,6)ANS(1)
      WRITE(6,7)(B(J),J=1,I)
      WRITE(6,8) I
      WRITE(6,9)
      SUM=ANS(4)
      WRITE(6,10) I,ANS(4),ANS(6),ANS(10),SUMIP
      NI=ANS(8)
      WRITE(6,11) NI,ANS(7),ANS(9)
      WRITE(6,12) NT,SUMSQ(MM)
      COE(1)=ANS(1)
      DO 160 J=1,I
160   COE(J+1)=B(J)
      LA=I
200   CONTINUE
C     TEST WHETHER PLOT IS REQUIRED OR NO)
210   IF(NPLOT) 100,100,220
220   NP3=N+N
      DO 230 I=1,N
      NP3=NP3+1
      P(NP3)=COE(1)
      L=I
      DO 230 J=1,LA
      P(NP3)=P(NP3)+X(L)*COE(J+1)
230   L=L+N
      N2=N
      L=N*M
      DO 240 I=1,N
      P(I)=X(I)
      N2=N2+1
      L=L+1
240   P(N2)=X(L)
      WRITE(6,3)
      WRITE(6,5) LA
      WRITE(6,14)
      NP2=N
      NP3=N+N
      DO 250 I=1,N
      NP2=NP2+1
      NP3=NP3+1
      RESID=P(NP2)-P(NP3)
250   WRITE(6,15) I,P(I),P(NP2),P(NP3),RESID
      CALL PLOT(LA,P,N,3,0,0)
      STOP
      END
      SUBROUTINE ORDER(M,R,NDEP,K,ISAVE,RX,RY)
      DIMENSION R(1),ISAVE(1),RX(1),RY(1)
```

```
      MM=0
      DO 130 J=1,K
      L2=ISAVE(J)
      IF(NDEP-L2)122,123,123
122   L=NDEP+(L2*L2-L2)/2
      GO TO 125
123   L=L2+(NDEP*NDEP-NDEP)/2
125   RY(J)=R(L)
      DO 130 I=1,K
      L1=ISAVE(I)
      IF(L1-L2)127,128,128
127   L=L1+(L2*L2-L2)/2
      GO TO 129
128   L=L2+(L1*L1-L1)/2
129   MM=MM+1
130   RX(MM)=R(L)
      ISAVE(K+1)=NDEP
      RETURN
      END
C
      SUBROUTINE MINV(A,N,D,L,M)
      DIMENSION A(1),L(1),M(1)
C     SEARCH FOR LARGEST ELEMENT
      D=1.0
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
10    IF(ABS(BIGA)-ABS(A(IJ))) 15,20,20
15    BIGA=A(IJ)
      L(K)=I
      M(K)=J
20    CONTINUE
C     INTERCHANGE ROWS
      J=L(K)
      IF(J-K) 35,35,25
25    KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
30    A(JI)=HOLD
```

```
C       INTERCHANGE COLUMNS
35      I=M(K)
        IF(I-K)45,45,38
38      JP=N*(I-1)
        DO 40 J=1,N
        JK=NK+J
        JI=JP+J
        HOLD=-A(JK)
        A(JK)=A(JI)
40      A(JI)=HOLD
C       DIVIDE COLUMNS BY MINUS PIVOT
45      IF(BIGA)48,46,48
46      D=0.0
        RETURN
48      DO 55 I=1,N
        IF(I-K)50,55,50
50      IK=NK+I
        A(IK)=A(IK)/(-BIGA)
55      CONTINUE
C       REDUCE MATRIX
        DO 65 I=1,N
        IK=NK+I
        HOLD=A(IK)
        IJ=I-N
        DO 65 J=1,N
        IJ=IJ+N
        IF(I-K)60,65,60
60      IF(J-K)62,65,62
62      KJ=IJ-I+K
        A(IJ)=HOLD*A(KJ)+A(IJ)
65      CONTINUE
C       DIVIDE ROW BY PIVOT
        KJ=K-N
        DO 75 J=1,N
        KJ=KJ+N
        IF(J-K)70,75,70
70      A(KJ)=A(KJ)/BIGA
75      CONTINUE
C       PRODUCT OF PIVOTS
        D=D*BIGA
C       REPLACE PIVOT BY RECIPROCAL
        A(KK)=1.0/BIGA
80      CONTINUE
C       FINAL ROW AND COLUMN INTERCHANGE
        K=N
100     K=(K-1)
        IF(K)150,150,105
105     I=L(K)
        IF(I-K)120,120,108
108     JQ=N*(K-1)
```

```
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
110   A(JI)=HOLD
120   J=M(K)
      IF(J-K)100,100,125
125   KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
130   A(JI)=HOLD
      GO TO 100
150   RETURN
      END
C
      SUBROUTINE MULTR(N,K,XBAR,STD,D,RX,RY,ISAVE,B,SB,T,ANS)
      DIMENSION XBAR(1),STD(1),RX(1),RY(1),D(1),ISAVE(1),
      1B(1),SB(1),T(1),ANS(1)
      MM=K+1
C     BETA WEIGHTS
      DO 100 J=1,K
100   B(J)=0.0
      DO 110 J=1,K
      L1=K*(J-1)
      DO 110 I=1,K
      L=L1+I
110   B(J)=B(J)+RY(I)*RX(L)
      RM=0.0
      BO=0.0
      L1=ISAVE(MM)
C     COEFFICIENTS OF DETERMINATION
      DO 120 I=1,K
      RM=RM+B(I)*RY(I)
C     REGRESSION COEFFICIENTS
      L=ISAVE(I)
      B(I)=B(I)*(STD(L1)/STD(L))
C     INTERCEPT
120   BO=BO+B(I)*XBAR(L)
      BO=XBAR(L1)-BO
C     SUM OF SQUARES ATTRIBUTABLE TO REGRESSION
      SSAR=RM*D(L1)
C     MULTIPLE CORRELATION COEFFICIENT
122   RM=SQRT(ABS(RM))
C     SUM OF SQUARES OF DEVIATIONS FROM REGRESSION
      SSDR=D(L1)-SSAR
```

```
C       VARIANCE OF ESSTIMATE
        FN=N-K-1
        SY=SSDR/FN
C       STANDARD DEVIATIONS OF REGRESSION COEFFICIENTS
        DO 130 J=1,K
        L1=K*(J-1)+J
        L=ISAVE(J)
125     SB(J)=SQRT(ABS((RX(L1)/D(L))*SY))
C       COMPUTED T-VALUES
130     T(J)=B(J)/SB(J)
C       STANDARD ERROR OF ESTIMATE
135     SY=SQRT(ABS(SY))
C       F VALUE
        FK=K
        SSARM=SSAR/FK
        SSDRM=SSDR/FN
        F=SSARM/SSDRM
        ANS(1)=BO
        ANS(2)=RM
        ANS(3)=SY
        ANS(4)=SSAR
        ANS(5)=FK
        ANS(6)=SSARM
        ANS(7)=SSDR
        ANS(8)=FN
        ANS(9)=SSDRM
        ANS(10)=F
        RETURN
        END
        SUBROUTINE PLOT(NO,A,N,M,NL,NS)
        DIMENSION OUT(101),YPR(11),ANG(9),A(300)
  1     FORMAT(1H1,60X,7H CHART ,I3//)
  2     FORMAT(1H ,F11.4,5X,101A1)
 3      FORMAT(1H )
  4     FORMAT(10H 123456789)
  5     FORMAT(10A1)
  7     FORMAT(1H ,16X,101H .
        1          .
        1          .)
  8     FORMAT(1H0,9X,11F10.4)
        NLL=NL
        IF(NS)16,16,10
C       SORT BASE VARIABLE IN ASCENDING ORDER
 10     DO 15 I=1,N
        DO 14 J=1,N
        IF(A(I)-A(J)) 14,14,11
 11     L=I-N
        LL=J-N
        DO 12 K=1,M
        L=L+N
```

```
         LL=LL+N
         F=A(L)
         A(L)=A(LL)
   12    A(LL)=F
   14    CONTINUE
   15    CONTINUE
C TEST NLL
   16    IF(NLL) 20,18,20
   18    NLL=50
   20    WRITE(6,1) NO
         REWIND 13
         WRITE(13,4)
         REWIND 13
         READ(13,5) BLANK,(ANG(I),I=1,9)
         REWIND 13
C  FIND SCALE FOR BASE VARIABLE
         XSCAL=(A(N)-A(1))/(FLOAT(NLL-1))
C  FIND SCALE FOR CROSS VARIABLE
         M1=N+1
         YMIN=A(M1)
         YMAX=YMIN
         M2=M*N
         DO 40 J=M1,M2
         IF(A(J)-YMIN) 28,26,26
   26    IF(A(J)-YMAX) 40,40,30
   28    YMIN=A(J)
         GOTO 40
   30    YMAX=A(J)
   40    CONTINUE
         YSCAL=(YMAX-YMIN)/100.0
C  FIND BASE VARIABLE PRINT POSITION
         XB=A(1)
         L=1
         MY=M-1
         I=1
   45    F=I-1
         XPR=XB+F*XSCAL
         IF(A(L)-XPR) 50,50,70
   50    DO 55 IX=1,101
   55    OUT(IX)=BLANK
         DO 60 J=1,MY
         LL=L+J*N
         JP=((A(LL)-YMIN)/YSCAL)+1.0
         OUT(JP)=ANG(J)
   60    CONTINUE
         WRITE(6,2)XPR,(OUT(IZ),IZ=1,101)
         L=L+1
         GOTO 80
   70    WRITE(6,3)
   80    I=I+1
```

```
      IF(I-NLL)45,84,86
 84   XPR=A(N)
      GOTO 50
 86   WRITE(6,7)
      YPR(1)=YMIN
      DO 90 KN=1,9
 90   YPR(KN+1)=YPR(KN)+YSCAL*10.0
      YPR(11)=YMAX
      WRITE(6,8)(YPR(IP),IP=1,11)
      RETURN
      END
      SUBROUTINE GDATA(N,M,X,XBAR,STD,D,SUMSQ)
      DIMENSION X(1),XBAR(1),STD(1),D(1),SUMSQ(1)
      IF(M-1) 105,105,90
 90   L1=0
      DO 100 I=2,M
      L1=L1+N
      DO 100 J=1,N
      L=L1+J
      K=L-N
100   X(L)=X(K)*X(J)
105   MM=M+1
      DF=N
      L=0
      DO 115 I=1,MM
      XBAR(I)=0.
      DO 110 J=1,N
      L=L+1
110   XBAR(I)=XBAR(I)+X(L)
115   XBAR(I)=XBAR(I)/DF
      DO 130 I=1,MM
130   STD(I)=0.
      L=((MM+1)*MM)/2
      DO 150 I=1,L
150   D(I)=0.
      DO 170 K=1,N
      L=0
      DO 170 J=1,MM
      L2=N*(J-1)+K
      T2=X(L2)-XBAR(J)
      STD(J)=STD(J)+T2
      DO 170 I=1,J
      L1=N*(I-1)+K
      T1=X(L1)-XBAR(I)
      L=L+1
170   D(L)=D(L)+T1*T2
      L=0
      DO 175 J=1,MM
      DO 175 I=1,J
      L=L+1
```

```
 175    D(L)=D(L)-STD(I)*STD(J)/DF
        L=0
        DO 180 I=1,MM
        L=L+I
        SUMSQ(I)=D(L)
 180    STD(I)=SQRT(ABS(D(L)))
C       CALCULATE CORRELATION COEFFICIENT
        L=0
        DO 190 J=1,MM
        DO 190 I=1,J
        L=L+1
 190    D(L)=D(L)/(STD(I)*STD(J))
C       CALCULATE STANDARD DEVIATION
        DF=SQRT(DF-1.0)
        DO 200 I=1,MM
 200    STD(I)=STD(I)/DF
        RETURN
        END
```

# APPENDIX II

## TEST INPUT

93 YEARS STAGE DATA AT BROACH
00093021

| | |
|---|---|
| 1. | 26.50 |
| 2. | 21.00 |
| 3. | 23.50 |
| 4. | 25.00 |
| 5. | 22.50 |
| 6. | 27.50 |
| 7. | 24.50 |
| 8. | 30.00 |
| 9. | 21.50 |
| 10. | 30.00 |
| 11. | 27.00 |
| 12. | 30.00 |
| 13. | 18.25 |
| 14. | 28.00 |
| 15. | 26.50 |
| 16. | 20.50 |
| 17. | 23.50 |
| 18. | 21.50 |
| 19. | 30.70 |
| 20. | 32.50 |
| 21. | 25.00 |
| 22. | 24.50 |
| 23. | 17.00 |
| 24. | 25.50 |
| 25. | 20.50 |
| 26. | 28.00 |
| 27. | 33.00 |
| 28. | 21.50 |
| 29. | 24.50 |
| 30. | 27.50 |
| 31. | 27.00 |
| 32. | 19.50 |
| 33. | 32.50 |
| 34. | 22.50 |
| 35. | 24.50 |
| 36. | 24.50 |
| 37. | 30.00 |
| 38. | 27.00 |
| 39. | 22.00 |
| 40. | 29.50 |
| 41. | 28.00 |
| 42. | 27.50 |
| 43. | 25.00 |
| 44. | 30.50 |
| 45. | 29.50 |
| 46. | 29.00 |
| 47. | 34.00 |
| 48. | 29.50 |

| | |
|---|---|
| 49. | 23.50 |
| 50. | 26.50 |
| 51. | 30.50 |
| 52. | 30.50 |
| 53. | 34.00 |
| 54. | 26.50 |
| 55. | 29.00 |
| 56. | 34.50 |
| 57. | 25.50 |
| 58. | 35.00 |
| 59. | 33.00 |
| 60. | 27.50 |
| 61. | 31.50 |
| 62. | 29.50 |
| 63. | 29.00 |
| 64. | 35.00 |
| 65. | 23.50 |
| 66. | 23.50 |
| 67. | 33.00 |
| 68. | 29.50 |
| 69. | 28.50 |
| 70. | 27.00 |
| 71. | 27.50 |
| 72. | 25.50 |
| 73. | 33.00 |
| 74. | 25.50 |
| 75. | 25.20 |
| 76. | 32.30 |
| 77. | 23.00 |
| 78. | 25.25 |
| 79. | 23.00 |
| 80. | 20.60 |
| 81. | 23.75 |
| 82. | 38.50 |
| 83. | 29.60 |
| 84. | 41.50 |
| 85. | 26.00 |
| 86. | 32.00 |
| 87. | 36.50 |
| 88. | 29.98 |
| 89. | 28.50 |
| 90. | 26.50 |
| 91. | 26.99 |
| 92. | 29.54 |
| 93. | 31.49 |

APPENDIX III

TEST OUTPUT

93 YEARS STAGE DATA AT BROACH
POLYNOMIAL REGRESSION......

NUMBER OF OBSERVASIONS 93

POLYNOMIAL REGRESSION OF DEGREE 1
INTERCEPT 24.6236

REGRESSION COEFFICIENTS
0.0611

0

ANALYSIS OF VARIENCE FOR 1DEGREE OF POLYNOMIAL

| 0 SOURCE OF VARIATION | DEGREE OF FREEDOM | SUM OF SQUARES | MEAN SQUARE | F VALUE | IMPROVEMENT INTERMS OF SUM OF SQUARES |
|---|---|---|---|---|---|
| 0DUE TO REGRESSION | 1 | 250.38 | 250.38 | 14.10 | 250.38 |
| DEVIAION ABOUT REGRESSION | 91 | 1615.39 | 17.75 | | |
| TOTAL | 92 | 1865.77 | | | |

POLYNOMIAL REGRESSION OF DEGREE 2
INTERCEPT 23.8397

REGRESSION COEFFICIENTS
0.1106 -0.0005

0

ANALYSIS OF VARIENCE FOR 2DEGREE OF POLYNOMIAL

| 0 SOURCE OF VARIATION | DEGREE OF FREEDOM | SUM OF SQUARES | MEAN SQUARE | F VALUE | IMPROVEMENT INTERMS OF SUM OF SQUARES |
|---|---|---|---|---|---|
| 0DUE TO REGRESSION | 2 | 261.09 | 130.55 | 7.32 | 10.71 |
| DEVIAION ABOUT REGRESSION | 90 | 1604.68 | 17.83 | | |
| TOTAL | 92 | 1865.77 | | | |

POLYNOMIAL REGRESSION......

POLYNOMIAL REGRESSION OF DEGREE 2

0

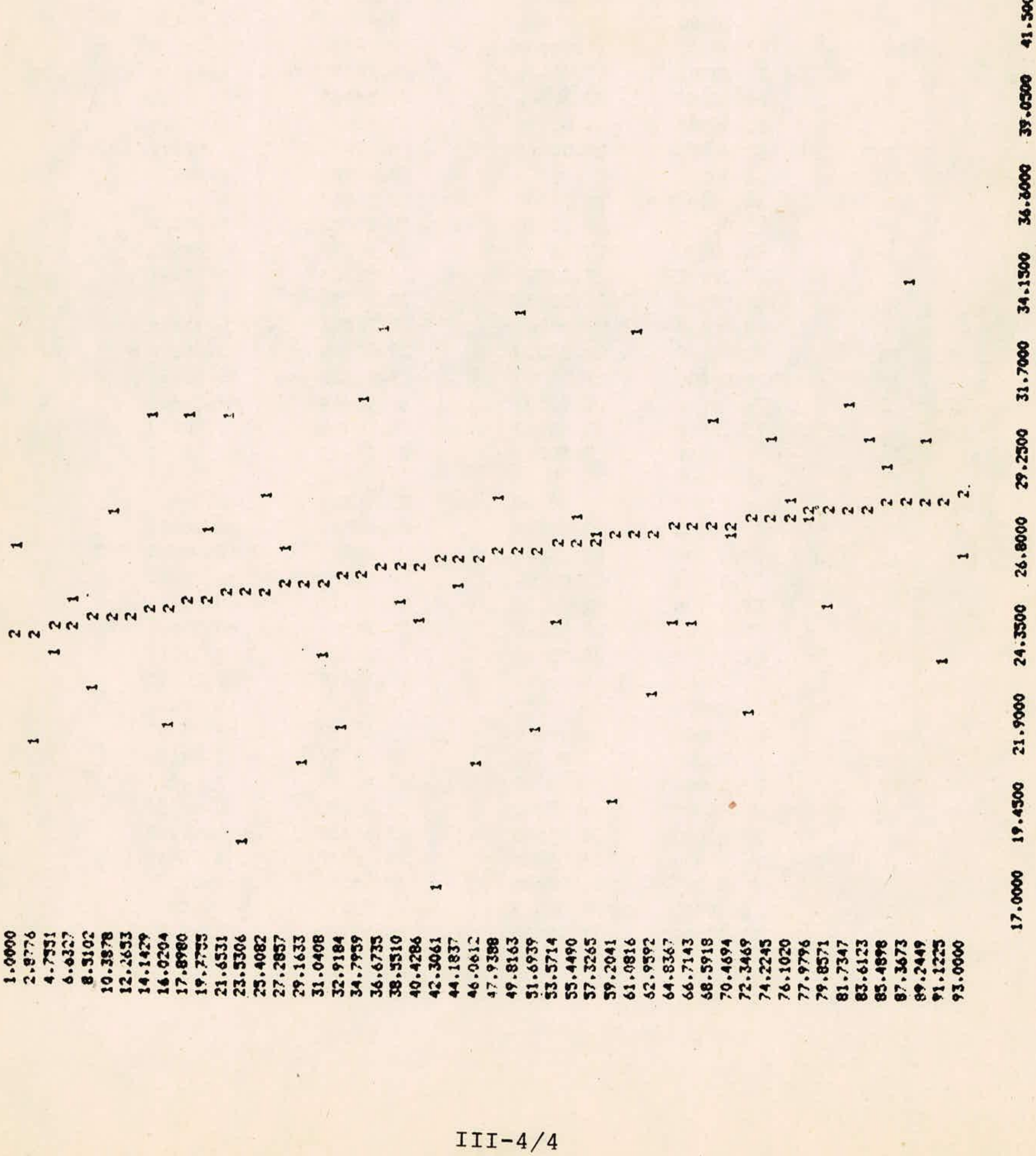
TABLE OF RESIDUALS

| OBSERVATION NO., | X VALUE | Y VALUE | Y ESTIMATE | RESIDUAL |
|---|---|---|---|---|
| 1 | 1.00000 | 26.50000 | 23.94983 | 2.55017 |
| 2 | 2.00000 | 21.00000 | 24.05888 | -3.05888 |

| | | | | |
|---|---|---|---|---|
| 3 | 3.00000 | 23.50000 | 24.16687 | -0.66687 |
| 4 | 4.00000 | 25.00000 | 24.27381 | 0.72619 |
| 5 | 5.00000 | 22.50000 | 24.37969 | -1.87969 |
| 6 | 6.00000 | 27.50000 | 24.48453 | 3.01547 |
| 7 | 7.00000 | 24.50000 | 24.58831 | -0.08831 |
| 8 | 8.00000 | 30.00000 | 24.69103 | 5.30897 |
| 9 | 9.00000 | 21.50000 | 24.79271 | -3.29271 |
| 10 | 10.00000 | 30.00000 | 24.89333 | 5.10667 |
| 11 | 11.00000 | 27.00000 | 24.99289 | 2.00711 |
| 12 | 12.00000 | 30.00000 | 25.09141 | 4.90859 |
| 13 | 13.00000 | 18.25000 | 25.18887 | -6.93887 |
| 14 | 14.00000 | 28.00000 | 25.28527 | 2.71473 |
| 15 | 15.00000 | 26.50000 | 25.38063 | 1.11937 |
| 16 | 16.00000 | 20.50000 | 25.47493 | -4.97493 |
| 17 | 17.00000 | 23.50000 | 25.56817 | -2.06817 |
| 18 | 18.00000 | 21.50000 | 25.66037 | -4.16037 |
| 19 | 19.00000 | 30.70000 | 25.75151 | 4.94849 |
| 20 | 20.00000 | 32.50000 | 25.84159 | 6.65841 |
| 21 | 21.00000 | 25.00000 | 25.93063 | -0.93063 |
| 22 | 22.00000 | 24.50000 | 26.01861 | -1.51861 |
| 23 | 23.00000 | 17.00000 | 26.10553 | -9.10553 |
| 24 | 24.00000 | 25.50000 | 26.19141 | -0.69141 |
| 25 | 25.00000 | 20.50000 | 26.27623 | -5.77623 |
| 26 | 26.00000 | 28.00000 | 26.35999 | 1.64001 |
| 27 | 27.00000 | 33.00000 | 26.44271 | 6.55729 |
| 28 | 28.00000 | 21.50000 | 26.52437 | -5.02437 |
| 29 | 29.00000 | 24.50000 | 26.60498 | -2.10498 |
| 30 | 30.00000 | 27.50000 | 26.68453 | 0.81547 |
| 31 | 31.00000 | 27.00000 | 26.76303 | 0.23697 |
| 32 | 32.00000 | 19.50000 | 26.84048 | -7.34048 |
| 33 | 33.00000 | 32.50000 | 26.91687 | 5.58313 |
| 34 | 34.00000 | 22.50000 | 26.99221 | -4.49221 |
| 35 | 35.00000 | 24.50000 | 27.06650 | -2.56650 |
| 36 | 36.00000 | 24.50000 | 27.13973 | -2.63973 |
| 37 | 37.00000 | 30.00000 | 27.21191 | 2.78809 |
| 38 | 38.00000 | 27.00000 | 27.28304 | -0.28304 |
| 39 | 39.00000 | 22.00000 | 27.35311 | -5.35311 |
| 40 | 40.00000 | 29.50000 | 27.42213 | 2.07787 |
| 41 | 41.00000 | 28.00000 | 27.49010 | 0.50990 |
| 42 | 42.00000 | 27.50000 | 27.55701 | -0.05701 |
| 43 | 43.00000 | 25.00000 | 27.62288 | -2.62288 |
| 44 | 44.00000 | 30.50000 | 27.68768 | 2.81232 |
| 45 | 45.00000 | 29.50000 | 27.75144 | 1.74856 |
| 46 | 46.00000 | 29.00000 | 27.81414 | 1.18586 |
| 47 | 47.00000 | 34.00000 | 27.87578 | 6.12422 |
| 48 | 48.00000 | 29.50000 | 27.93638 | 1.56362 |
| 49 | 49.00000 | 23.50000 | 27.99592 | -4.49592 |
| 50 | 50.00000 | 26.50000 | 28.05441 | -1.55441 |
| 51 | 51.00000 | 30.50000 | 28.11184 | 2.38816 |
| 52 | 52.00000 | 30.50000 | 28.16822 | 2.33178 |

| | | | | |
|---|---|---|---|---|
| 53 | 53.00000 | 34.00000 | 28.22355 | 5.77645 |
| 54 | 54.00000 | 26.50000 | 28.27782 | -1.77782 |
| 55 | 55.00000 | 29.00000 | 28.33104 | 0.66896 |
| 56 | 56.00000 | 34.50000 | 28.38321 | 6.11679 |
| 57 | 57.00000 | 25.50000 | 28.43432 | -2.93432 |
| 58 | 58.00000 | 35.00000 | 28.48438 | 6.51562 |
| 59 | 59.00000 | 33.00000 | 28.53339 | 4.46661 |
| 60 | 60.00000 | 27.50000 | 28.58135 | -1.08135 |
| 61 | 61.00000 | 31.50000 | 28.62825 | 2.87175 |
| 62 | 62.00000 | 29.50000 | 28.67409 | 0.82591 |
| 63 | 63.00000 | 29.00000 | 28.71889 | 0.28111 |
| 64 | 64.00000 | 35.00000 | 28.76263 | 6.23737 |
| 65 | 65.00000 | 23.50000 | 28.80532 | -5.30532 |
| 66 | 66.00000 | 23.50000 | 28.84695 | -5.34695 |
| 67 | 67.00000 | 33.00000 | 28.88753 | 4.11247 |
| 68 | 68.00000 | 29.50000 | 28.92706 | 0.57294 |
| 69 | 69.00000 | 28.50000 | 28.96553 | -0.46553 |
| 70 | 70.00000 | 27.00000 | 29.00295 | -2.00295 |
| 71 | 71.00000 | 27.50000 | 29.03932 | -1.53932 |
| 72 | 72.00000 | 25.50000 | 29.07464 | -3.57464 |
| 73 | 73.00000 | 33.00000 | 29.10890 | 3.89110 |
| 74 | 74.00000 | 25.50000 | 29.14211 | -3.64211 |
| 75 | 75.00000 | 25.20000 | 29.17426 | -3.97426 |
| 76 | 76.00000 | 32.30000 | 29.20536 | 3.09464 |
| 77 | 77.00000 | 23.00000 | 29.23541 | -6.23541 |
| 78 | 78.00000 | 25.25000 | 29.26440 | -4.01440 |
| 79 | 79.00000 | 23.00000 | 29.29235 | -6.29235 |
| 80 | 80.00000 | 20.60000 | 29.31923 | -8.71923 |
| 81 | 81.00000 | 23.75000 | 29.34507 | -5.59507 |
| 82 | 82.00000 | 38.50000 | 29.36985 | 9.13015 |
| 83 | 83.00000 | 29.60000 | 29.39358 | 0.20642 |
| 84 | 84.00000 | 41.50000 | 29.41625 | 12.08375 |
| 85 | 85.00000 | 26.00000 | 29.43787 | -3.43787 |
| 86 | 86.00000 | 32.00000 | 29.45844 | 2.54156 |
| 87 | 87.00000 | 36.50000 | 29.47796 | 7.02204 |
| 88 | 88.00000 | 29.98000 | 29.49642 | 0.48358 |
| 89 | 89.00000 | 28.50000 | 29.51383 | -1.01383 |
| 90 | 90.00000 | 26.50000 | 29.53018 | -3.03018 |
| 91 | 91.00000 | 26.99000 | 29.54548 | -2.55548 |
| 92 | 92.00000 | 29.54000 | 29.55973 | -0.01973 |
| 93 | 93.00000 | 31.49000 | 29.57292 | 1.91708 |

CHART 2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