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MODELLING OF FLOW IN THE CANAL DUE TO BREACH OF BANK PART-I

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PREFACE

Floods in India cause frequent breaching of canals throughout India. Due to the high seasonal variability of rainfall and due to cyclone in coastal India flood discharges cause many occurrence of breaches causing loss of life and severe damage to property.

Study of breach of a canal is relatively a new study as compared to dam breach, which is a well-researched field. The analysis of the problem is complicated due to the dynamics of flow within the canal. One-dimensional analysis of canal breach has been carried out in NIH. The limitations of the above model due to the one-dimensional analysis of the problem is hoped to be overcome in two or three dimensions analysis.

Wavelet element method is a new approach to modelling. It is a fast developing field with promise of exciting and varied applications in engineering and also hydrology. The analysis and the framework for the work of simulation have been carried out in this report and the work of simulation has to be carried out on this basis. This work is a pioneering work in Hydrology.

This study has been carried out by Dr.C.Rangaraj, Scientist `B', Flood Studies Div ision, under the supervision of Dr. S. K. Mishra, Scientist `E', Head, Flood Studies Division. With enhanced capabilities of computation with higher speeds and bigger storage on computers it is hoped that better simulation using higher dimensional models will realistically model the problem of canal breach.

DIRECTOR

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LIST OF SYMBOLS

A^*	Adjoint of A
C	Set of complex numbers
C^N	Set of all ordered N-tuples of complex numbers
$\mathbf{C}_M^N$	A N-row M column matrix of complex numbers
C^R	Infinite dimensional complex vector space possibly including non-linear, functions f:R -> C
F	Transform
FT	Fourier transform
FFT	Fast Fourier transform
f	Frequency
I	Identity matrix with diagonal elements 1 and off-diagonal elements zero
K	Kernel of transformation
$L^2(R)$.	Space of square-integrable complex valued functions on R
Me	Mexican hat wavelet
Mo	Morlet wavelet
R	Set of all real numbers
R^N	Set of all ordered N-tuples of real numbers
$\mathbf{R}_M^N$	A N-row M column matrix of real numbers
STFT	Short Time Fourier Transforms
supp	Support
t	Time
V	Independent variable like time or length
WFT	Windowed Fourier transform
α_j^i	Individual element of the Matrix in the i^{th} row and j^{th} column
γ	Specific weight
$\aleph$ and $\Im$	Hilbert space
$\lambda^2(Z)$.	Space of square summable complex sequences
$\lambda_1\lambda_1....\lambda_N$	Eigen values of matrix
η	set of integers = {1,2,3......N}

ρ Density

ν Kinematic viscosity

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ABSTRACT

Canal breach modelling is of great significance in open channel hydraulics. A limited amount of literature parallel to dam break analysis is available. In this report, a methodology is suggested based on wavelet element method, a recent breakthrough.

Wavelet element method and Wavelet transforms is a new field (The concepts were developed by mathematicians in the last 10 years). The subject is expanding at a tremendous rate in varied directions. It is now that the concepts are being used in engineering. The initial applications are in signal processing and in data compression. The solution offered by wavelet transforms to the FBI data storage problem of storing fingerprints has dramatically shown the usefulness of wavelet analysis. The theoretical basis of dual and bi-orthogonal systems is presented.

Wavelet analysis is a method to separate the data into different frequencies and systematically study each component with a resolution matched to scale. They have distinct advantages over the Fourier methods. Wavelet method of analysis arose to remedy the shortcomings of Fourier analysis.

Wavelet element method combines the spectral methods with wavelet transforms. A bi-orthogonal wavelet system is obtained. Equations of motion and the boundary conditions require to be adapted to the multi-resolution paradigm. In this report, the wavelet element method is developed for hydrological applications purposes, in general, and modelling a canal breach in particular. To this end the fundamental concepts available elsewhere are reviewed, synthesised and discussed comprehensively and canal breach method is described in steps for future pragmatic application. The major finding of the complex mathematical development is that it can prove to be vital in hydraulic and hydrological applications in future.

INTRODUCTION

Study of breach of canal occurs frequently in several parts of the country during floods. A field study of the effects of cyclone was undertaken in Orissa revealed that canal breach multiplied the damaged caused by cyclone. A one-dimensional study of canal breach was under taken in a recent report by National Institute of Hydrology. However, one dimensional modelling has severe limitation in that the velocity, pressure distributions and consequently the extent of diversion of flow through the breach remains to be determined. Further the effect of unsteady flow into the canal needs to be simulated. Research in this area is not as advanced as the research on dam break floods have been. However, the study of the phenomena of canal breach is relatively a new research field and needs attention by hydraulicians and hydrologists. A two or three dimensional model requires efficient method of computation.

The possibility of running full turbulent simulations at high Reynolds numbers is at present beyond the reach of present methods of simulation. The spectral method provides a very accurate numerical filter so that very high wave number energy is retained in the computation, so that it can be calculated as turbulent energy.

In hydraulics spectral methods (for example using sin and cos functions as the basis set within a Galerkin approach) were rarely attempted due to a major impediment of matrix `remaining full' in a step prior to the solution. Quoting A.M.Davis (Coastal, Estuarial and Harbour engineers reference book. --1993 from 'super three dimensional models' in the article Quasi - three-dimensional modelling using mixed finite-difference and spectral models --- "By using the Legendre polynomials the one of the matrix to be derived from the flow equation remains full". The problem in electronic and electrical engineering was to a certain extent solved by the use of 'Fast Fourier Transforms', subsequent to which the problems could be solved millions of times faster. Wavelets are a fundamental breakthrough. To explain in a very simplified language --- wavelets are a new kind of sin and cos functions generated, which do not have the infinitely long tail.

The key concept in wavelet analysis is analysis according to scale. The overall features are identified and later the details are analysed -- which is somewhat similar to how the human vision function. It decomposes complicated signals into small number of elementary waveforms which are localised both in time and frequency. They are particularly adapted to non-stationary functions -- that is those functions whose frequency content changes with time.

Concepts of Fourier Transforms (different from Fourier Series but related to it) and Wavelet transform are closely related. We seek every time to define the corresponding concept in Wavelet Transform to that which occurs in Fourier Transforms. Transforms are either used to solve problems intractable in the 'natural domain (time or length(x)) or to obtain further information from a natural data. For example the time series of rainfall, could be better understood in terms of the frequency and its temporal location of it. Frequency content of a function reveals another dimension of the data. For example a simple piece of music has intensity or volume (amplitude), times at which it occur and the tone (function of frequency).

One more dimension of information in Windowed Fourier Transforms and Wavelet transform. In a sense they convert a function of one variable into two. The catch is that the independent variable say time and frequency are interrelated --- the frequency cannot be measured instantaneously. The precise measurement of frequency and time is incompatible. Sharp localisation in time and frequency are mutually exclusive. Contrary to popular opinion Heiseberg's uncertainty principle is not restricted to quantum mechanics -- it is a general property of functions.

Wavelet element method combines the wavelet ideas with the Galerkin formulation to solve differential equations. The Matrix which results is full when the spectral method using Fourier transform is used to model the flow equations. Due to the compact support when the wavelet basis functions are used the matrix which results can be expected to be sparse.

WAVELETS

A transform represents a function in another domain. Many times the new domain has a physical meaning. The Fourier transform captures the frequency information in a function.

Kernel of the transformation:

All transforms could be represented in the following form:

$$F(w) = \int_a^b K(x,w) f(x) dx \qquad (1)$$

where K(x,w) is a function fixed for a transform. The kernel of the Fourier Transform pair is

$$K(t,f) = e^{\pm i2\pi ft} = \cos(2\pi ft) \pm iSin(2\pi ft) \text{ with a = -}\infty \text{ and b = } \infty \qquad (2)$$

The kernel of the wavelet transform changes with the wavelet chosen.

One of the limitation of Fourier series is that it is unable to distinguish stationary and non stationary function. Stationary function is a function whose frequency component does not change with the independent variable. Consider the functions(Figs.1&2):

$$\left.\begin{aligned} 1)y_1(x) &= \cos(\pi x) + \cos(2\pi x) \quad 0 \le x \le 20 \\ 2)y_2(x) &= \begin{cases} \cos(\pi x) & 0 \le x \le 10 \\ \cos(2\pi x) & 10 \le x \le 20 \end{cases} \end{aligned}\right\} \qquad (3)$$

Fourier Transform are distinctly different from Fourier series. The fourier transform of y_1 and y_2

$$F(f) = \int_{-\infty}^{\infty} y(x)\, e^{-i2\pi ft} dx \qquad (4)$$

will give the existence of particular frequencies but not their location in space/time (henceforth called independent variable or V). If a function contains two frequencies

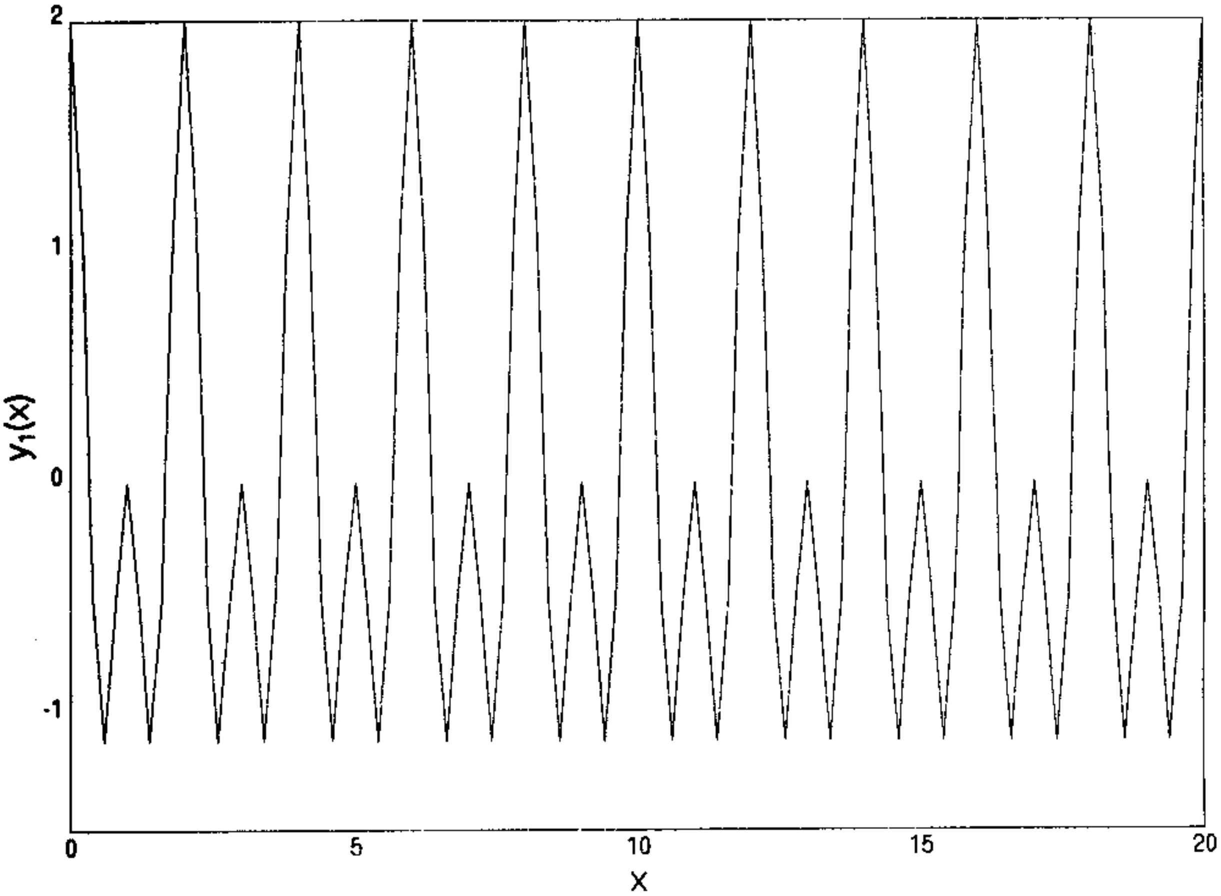

Fig.1 $y_1(x) = \cos(\pi x)+\cos(2\pi x)$ ...0<x<20

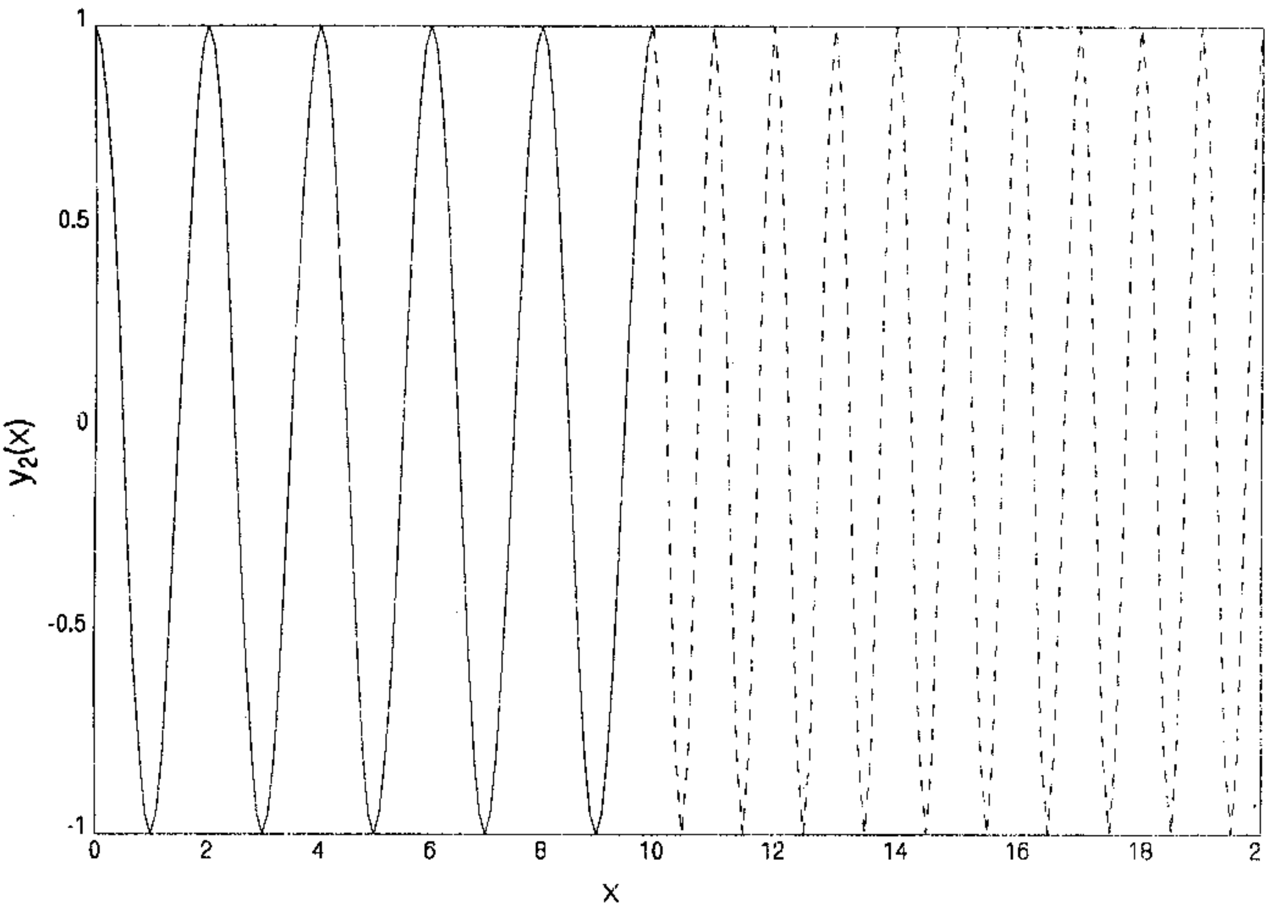

Fig.2 $y_2(x) = \cos(\pi x)$---$0<x<10$ and $\cos(2\pi x)$---$10<x<20$

separated over V Fourier transform does not analyse the response for input containing frequencies separated over V. In a Fourier transform the function to be analysed is assumed to be stationary. i.e, all frequency components exist at all V's. An independent variable-frequency representation of the function is that which is necessary.

Windows were tried in Short Time Fourier Transforms (STFT) initially to remedy the short coming of Fourier transforms. The 'window' function is zero outside a range of independent variable (example, g(u) = (1+cos(π u), -1 $\leq u \leq 1$; and g(u) is zero elsewhere) weighs the function over a specific location of the independent variable for transformation. A function $g_{\omega,t}(u) = e^{i2\pi wu} g(u-t)$ parameterized by all frequencies w and independent variable t form a kind of a basis function for a Hilbert space i.e., $L^2(\mathbf{R})$. Defining a function of both the variables, frequency and independent variable, $\tilde{f}(\omega,t) = \hat{f}_t(\omega) = \int_{-\infty}^{\infty} e^{-i2\pi\omega u} f_t(u)\, du$ where $f_t(u) = \bar{g}(u-t) f(u)$ and $\bar{g}(u-t)$ is the complex conjugate of g(u) shifted by time t; $\tilde{f}(\omega,t)$ is well defined even though f(u) may not be well defined at individual points(f(u) could be modified on a set of measure zero). It is because the function $\tilde{f}(\omega,t)$ written as the inner product $< g_{\omega,t}, f >$ is a kind of correlation between $g_{\omega,t}(u)$ and $f(u)$ and correlation is generally well behaved than either one of the functions. Further, by Schwarz inequality $\tilde{f}(\omega,t)$ = $|< g_{\omega,t}, f >|$ $\leq \| g_{\omega,t} \| \|f\|$ --- $\tilde{f}(\omega,t)$ is bounded. Therefore the windowed transform behaves better than the function itself or the Fourier transforms in the mathematical sense. Further Wavelet Transforms (WT) were developed to overcome the resolution problems associated with STFT. Wavelet transforms provide a independent variable-frequency representation of a function. The concept corresponding to the phase has to be changed to that of 'power spectrum' as the natural data being dealt with is relatively random or with 'noise'. Here the concept of phase loses all meaning. It is the form of square of the Fourier transform.

Admissibility criteria:

The admissibility criteria requires that the wavelet must be a small wave. The wavelet function must wiggle around the independent variable axis. For the wavelet transform to have its inverse and Parsevals theorem (See appendix 1) to be applicable is that it is necessary that it should satisfy the admissibility criteria:

$$\int_{-\infty}^{\infty} wavelet\ function = 0 \tag{5}$$

It follows from the requirement that $0 < \int_{-\infty}^{\infty} \frac{d\xi}{\xi} |\hat{\psi}|^2 < \infty$ It is seen that the local mean value of the function is automatically subtracted in the wavelet transforms.

Need for wavelet transforms:

Problem of resolution does not exist for a Fourier Transform. Neither frequency resolution nor V resolution exists for FT. Exact values of frequencies are known and the value of the function is known at every instant of V . However the cross resolution, the V resolution in the frequency domain, and the frequency resolution in the V domain are nil, since no information about them is possible. Faultless frequency resolution in the FT is obtained because the kernel which can be thought of as a window, which is integrated for all V from minus infinity to plus infinity. In STFT, the window is of finite length and it covers only a portion of the function, which makes frequency resolution poorer. Exact frequency components that exist in the function are not known. Information of the frequency is spread over a band. If the width of the window is extended from -∞ to +∞ it reduces to a Fourier transform, which gives perfect frequency resolution, but not the location of the frequency in the independent variable axis.

A function is said to be stationary if its frequency content does not change with time. In Fourier transform the function is implicitly assumed to be stationary for all times. In order to obtain the stationarity, a short enough window is required, in which the function can be assumed to be stationary. As the window is made narrow V resolution gets better but the frequency resolution gets worse.

Multi-resolution analysis:

Before 1986 the analysis (computation of f) would have to be made directly by integration. Multiresolution analysis is a radically new method for performing analysis and synthesis. This method is completely recursive and ideal for computations. Refer section "Example construction of wavelet". Beginning with a function sampled at regular intervals of $\Delta t = \tau > 0$, $f^0 = \{f_n^0\}_{n\in Z}$. f^0 is split into blurred out version at coarser scale $\Delta t = 2\tau$ and detail d^1 at scale $\Delta t = \tau$. Each detail can be written as superposition of wavelets and the mother wavelet is determined by the blurring process. After N iteration the original function can be written as $f^0 = f^N + d^1 + d^2 + ...+ d^N$. Wavelets use narrow window for higher frequencies and wide window for low frequencies resulting in a adequate independent variable resolution at high frequencies and adequate frequency resolution at low frequencies. Multiresolution analysis begins with a scaling function from which the wavelets are determined. Table 1 gives the summary of frequency and independent variable resolution characteristics of windowed Fourier transforms and wavelets. It should however be noted that the process of fixing the size of window is automatic in the algorithm itself. The unexpected existance of such wavelets, which form an orthonormal basis is the reason for its current popularity. Quoting M.B. Abbot and D.R.Basco (1) who compares the process of good numerical simulation with radio amplifier with good frequency response: "...... Our aim in computational fluid dynamics is to design numerical techniques that reproduces as many as possible of these `sounds'...."

TABLE 1- SUMMARY OF FREQUENCY AND INDEPENDENT VARIABLE RESOLUTION CHARACTERISTICS OF WINDOWED FOURIER TRANSFORMS AND WAVELETS

Size of window	V resolution	Frequency resolution	Frequencies analysed in Windowed Fourier Transforms	Frequencies analysed in wavelets
Narrow	Adequate	Inadequate	Single size of window is used for all frequencies for a analysis	Higher
Wide	inadequate	Adequate		Lower

Examples of wavelets:

Mexican Hat Wavelet

It can be expressed as

$$Me(z) = (z^2 - 1)\, e^{-z^2/2} \quad (6)$$

ɔr with its energy normalised

$$\frac{1}{\sqrt{2\pi}\sigma^3}\{e^{-t^2/2\sigma^2}(\frac{t^2}{\sigma^2} - 1)\} \quad (7)$$

The Morlet wavelet

$$\text{Mo(t)} = e^{-t^2/2\sigma}(\cos at + i \sin at) \quad (8)$$

σ is a factor which determines the size of the window.

WAVELET ANALYSIS AND SYNTHESIS

Wavelet Analysis:

Wavelets analysis is called multi-resolution analysis. Central to the analysis is the concepts of operator theory, Bi-orthogonality, dual spaces and an understanding of Hilbert spaces. Powerful methods of linear algebra can be extended to operators and functions by this understanding.

Scale Support Translation and Resolution:

The concept of scale is the starting point in generation of wavelet transforms. Mathematically if the independent variable is multiplied with a constant, the function is stretched or shrunk. In wavelet analysis since the scale parameter frequently appears in the denominator the opposite is true in all such cases. It is inverse of frequency. Since the wavelet itself is not analysed but a function is analysed using a wavelet the concept of scale is quite natural. Acceptable scale resolution means unacceptable frequency resolution and vice versa.

Support refers to the size of the transforming window in comparison with the `window' of a Fourier transform which is infinite. If a window vanishes outside of the finite interval [a,b] it is then called `compactly supported' and denoted by supp[a,b]. Usually it refers to the base width of the window.

Translation refers to the location of the window. It has the same units as V .

Resolution means to investigate which spectral component exists at any given interval of V WT allows for variable resolution. High frequency component in a function is located with a narrow window. A coarse view is taken and later finer scales are then chosen. Each is translated over the function and transformed.

The process of wavelet transform can be expressed as

$$WT_y^{\psi}(\tau,s) = \int y(t)\psi^*(\frac{t-\tau}{s})\,\mathrm{dt} \quad (9)$$

where, a matrix of the transformed function is obtained with one direction corresponding to the translation with a given scale and the other due to change of scale.

Wavelet synthesis:

When the admissibility criteria is satisfied synthesis becomes possible. Generally the bases will be orthogonal, bi-orthogonal or will be of type 'frames'. Bi-orthogonal basis refers to the condition that two sets of basis are orthogonal to each other but not to themselves. If for a problem orthogonal basis cannot be determined then bi-orthogonal basis are used. Even that failing frames are used. However orthonormal basis allow easy analysis and synthesis.

The inverse transform of a continuous wavelet transform is

$$y(t) = \frac{1}{C_\psi^2}\iint_{s,\tau}\Psi(\tau,s)\psi(\frac{t-\tau}{s})d\tau ds \quad (10)$$

DISCREET WAVELET TRANSFORM

Sampling theorem:

The sampling theorem states that, under certain conditions, it is possible to recover the intervening values of a sampled series accurately. The sample set is fully equivalent to the original set provided the function is band limited.

If function f(x) whose Fourier transform is zero where $|s| > s_c$, it is band limited. With any waveform there is always a frequency beyond which the spectral contributions are negligible. Band limited functions have the property they are fully specified by values spaced at equal intervals not exceeding $0.5/s_c$.

$$\tau^{-1} f(x) 111(\frac{x}{\tau}) = \sum_n f(n\tau)\,\delta(x - n\tau) \tag{11}$$

Information of f(x) is retained only at the sampling points at $x = n\tau$.

It is proposed to reconstruct the function from the sampled points ($f(x)111(x/\tau)$) when the sampling theorem is proved. Since, $111(x/\tau) \supset \tau 111(\tau s)$ it corresponds to a row of impulses at spacing $1/\tau$. By the replicating property of 111 function with a function (function appears in replica at unit intervals along the entire length of axis)

$$\overline{f(x)111(x/\tau)} = \tau\, 111(\tau s) * F(s) \tag{12}$$

F(s) is recoverable (If there is no overlap with the neighbouring transform, which is true for a band limited function and $1/\tau > 2s_c$ which in turn implies that the sampling interval $\tau < (1/2)\,(1/s_c)$) by multiplying $\tau 111(\tau s) * F(s)$ by $11(s/2s_c)$. The sampling interval should be lesser than the half the period of sinusoid of frequency s_c. Hence, f(x) is obtained as the inverse transform of this product. Critical sampling is when $\tau = (1/2)\,(1/s_c)$, when,

$$111(2s_c x) f(x) \supset (\frac{1}{2s_c})\, 111(\frac{2}{2s_c}) * F(s) \tag{13}$$

Filtering in the spectrum corresponds to interpolation in the function domain. Since recovery involves the multiplication of the transform by $11(s/2s_c)$ the equivalent operation in the function domain is convolution with the function $2s_c$ sinc ($2s_c$) which will yeild f(x) from the series of impulses $111(x/\tau)f(x)$. The process involves taking a serial product. When the interpolation process needs to be repeated the values of sinc x suitable spaced proves to be beneficial.

Rectangular filtering:

Assume that the critical sampling interval $\tau^{-1} = 1/(2*s_c) = 1$ when $s_c = 1/2$, if it is required to remove spectral components exceeding the limit s_c, it corresponds to multiplying the transform by 11(s). Numerically it will be an approximation represented by the summation

$$\sum\nolimits_{\tau} = f(x)*[111(\frac{x}{\tau})\text{sinc } x] \tag{14}$$

$$\overline{\sum\nolimits_{\tau}} = F(s)*[\tau 111(\tau s)*11(s)] \tag{15}$$

It is important to determine the coarseness of the tabulation interval allowing to sufficiently approximate the desired integral f(x)*sinc x (A function whose transform is cut off). Trying $\tau = 1$, $\Sigma_1 = f(x)$; there has been no filtering. Trying $\tau = 1/2$

$$\sum\nolimits_{1/2} = f(x)*[111(2x)\text{sinc } x] \tag{16}$$

$$\overline{\sum\nolimits_{1/2}} = F(s)*[\frac{1}{2}111(\frac{s}{2})*11(s)] \tag{17}$$

In other words a band limited function is a function, f(t), which has no spectral components beyond a frequency B Hz; that is, F(s) = 0 for |s| > 2B. The sampling theorem states that a real function, f(t), which is band limited to B Hz can be reconstructed without error from samples taken uniformly at a rate R > 2B samples per second. This minimum sampling frequency, s = 2B Hz, is called the Nyquist rate or the Nyquist frequency. The corresponding sampling interval, T = 1/2B (where t = nT), is called the Nyquist interval. A function band limited to B Hz which is sampled at less than the Nyquist frequency of 2B, i.e., which was sampled at an interval T > 1/2B, is said to be under sampled. With the Nyquist sampling rate in the V scale plane the frequency of sampling (which allows reconstruction of the original function) is given by the relation $N_2S_2 = N_1S_1$ where N's are the sampling rate and S's are the scales.

It should be noted at this time, however, that the discretization can be done in any way without any restriction as far as the analysis of the function is concerned. If synthesis is not required, even the Nyquist criteria does not need to be satisfied. The restrictions on the discretization and the sampling rate become important if, and only if, the function reconstruction is desired. Nyquist's sampling rate is the minimum sampling rate that allows the original continuous V function to be reconstructed from its discrete samples.

The distinction between the Fourier transform, Fourier series and discrete Fourier transform should be appreciated. Similarly there are continuous wavelet transform, semi-discrete wavelet transforms and discrete wavelet transforms. The critical point to be noticed is that the sampling rate is automatically adjusted to the scale in discrete wavelet transform.

A V -scale representation of a digital function is obtained using digital filtering techniques. CWT is a correlation between a wavelet at different scales and the function with the scale (or the frequency) being used as a measure of similarity. The continuous wavelet transform was computed by changing the scale of the analysis window, shifting the window in V , multiplying by the function, and integrating over all V's. In the discrete case, filters of different cutoff frequencies are used to analyze the function at different scales. The function is passed through a series of high pass filters to analyze the high frequencies, and it is passed through a series of low pass filters to analyze the low frequencies.

The resolution of the function, which is a measure of the amount of detail information in the function, is changed by the filtering operations, and the scale is changed by up-sampling and down-sampling (sub-sampling) operations. Sub-sampling a function corresponds to reducing the sampling rate, or removing some of the samples of the function.

For example, sub-sampling by two refers to dropping every other sample of the function. Sub-sampling by a factor n reduces the number of samples in the function n times.

Up-sampling a function corresponds to increasing the sampling rate of a function by adding new samples to the function. For example, up-sampling by two refers to adding a new sample, usually a zero or an interpolated value, between every two samples of the function. Up-sampling a function by a factor of n increases the number of samples in the function by a factor of n.

DWT coefficients are usually sampled from the CWT on a dyadic grid, i.e., s0 = 2 and t 0 = 1, yielding s=2j and t =k*2j. The procedure starts with passing this function (sequence) through a half band digital low pass filter with impulse response h[n]. Filtering a function corresponds to the mathematical operation of convolution of the function with the impulse response of the filter.

A half band low pass filter removes all frequencies that are above half of the highest frequency in the function. For example, if a function has a maximum of 1000 Hz component, then half band low pass filtering removes all the frequencies above 500 Hz. The unit of frequency is of particular importance at this time. In discrete functions, frequency is expressed in terms of radians. Accordingly, the sampling frequency of the function is equal to 2p radians in terms of radial frequency. Therefore, the highest frequency component that exists in a function will be p radians, if the function is sampled at Nyquist's rate (which is twice the maximum frequency that exists in the function); that is, the Nyquist's rate corresponds to p rad/s in the discrete frequency domain. Therefore using Hz is not appropriate for discrete functions. However, Hz is used whenever it is needed to clarify a discussion, since it is very common to think of frequency in terms of Hz. It should always be remembered that the unit of frequency for discrete time functions is radians. After passing the function through a half band low pass filter, half of the samples can be eliminated according to the Nyquist's rule, since the function now has a highest frequency of p/2 radians instead of p radians. Simply discarding every other sample will subsample the function by two, and the function will then have half the number of points. The scale of the function is now doubled. Low pass filtering removes the high frequency information, but leaves the scale unchanged. Only the sub-sampling process changes the scale. Resolution, on the other hand, is related to the amount of information in the function, and therefore, it is affected by the filtering operations. Half band low pass filtering removes half of the frequencies, which can be interpreted as losing half of the information. Therefore, the resolution is halved after the filtering operation. Sub-sampling operation after filtering does not affect the resolution, since removing half of the spectral components from the function makes half the number of samples redundant. Half the samples can be discarded without any loss of information. The function is then sub-sampled by 2 since half of the number of samples are redundant. This doubles the scale.

EXAMPLE CONSTRUCTION OF A WAVELET

For each nonnegative integer j let v^j be the vector space of piecewise constant functions on [0,1) with possible breaks at $\frac{1}{2^j}, \frac{2}{2^j} \dots \frac{2^j - 1}{2^j}$. Then the 2^j functions ϕ_i^j defined by $\phi(2^j\text{-}i)$ $0 \le i \le 2^j - 1$ form a basis for v^j. We get an infinite ascending chain of vector spaces $v^0 \subset v^1 \subset \dots\dots v^{j+1} \subset \dots\dots$ each of which is an inner product space with respect to the inner product $<f,g> = \int_0^1 f(t)\, g(t) dt$. The wavelet space ω^j is then defined to be the orthogonal complement of v^j in v^{j+1}. The functions $\chi_i^j(x) = \chi(2^j - i)$ for $0 \le i \le 2^j - 1$ form a basis for ω^j. For any j, $v^j = v^0 \oplus \omega^0 \oplus \omega^1 \dots\dots \oplus \omega^{j-1}$ (Orthogonal direct sum decomposition)

$$\phi = \begin{cases} 1 \ \text{on}[0,1) \\ 0 \ \text{Elsewhere} \end{cases} \tag{18}$$

Consider the histogram shown in Fig.3 on top. The dyadically scaled and translated scaling function $\phi_i^2 = \phi(2^2x\text{- }i)$. The mother wavelet(called Harr wavelet)

$$\chi(x) = \begin{cases} 1 \ \text{on}[0,\frac{1}{2}) \\ -1 \ \text{on}[\frac{1}{2},1) \\ 0 \ \text{Elsewhere} \end{cases} \tag{19}$$

The wavelet basis functions are (Fig.3)

$$\chi_i^2 = \chi(2^2x\text{- }i). \tag{20}$$

Example of hydrological data compression:

Let Φ be an NxN matrix. A transform operation gives

$$\Psi \ = \ W^T \Phi W \tag{21}$$

It is possible to obtain a matrix using wavelets which produces a transform matrix Ψ that is sparse and with most of its large magnitude elements concentrated in a small region of Ψ. It is "energy compaction" by transformation. In the example given below 37 units of input data and 25 units of transformed data (with threshold chosen to be zero) is

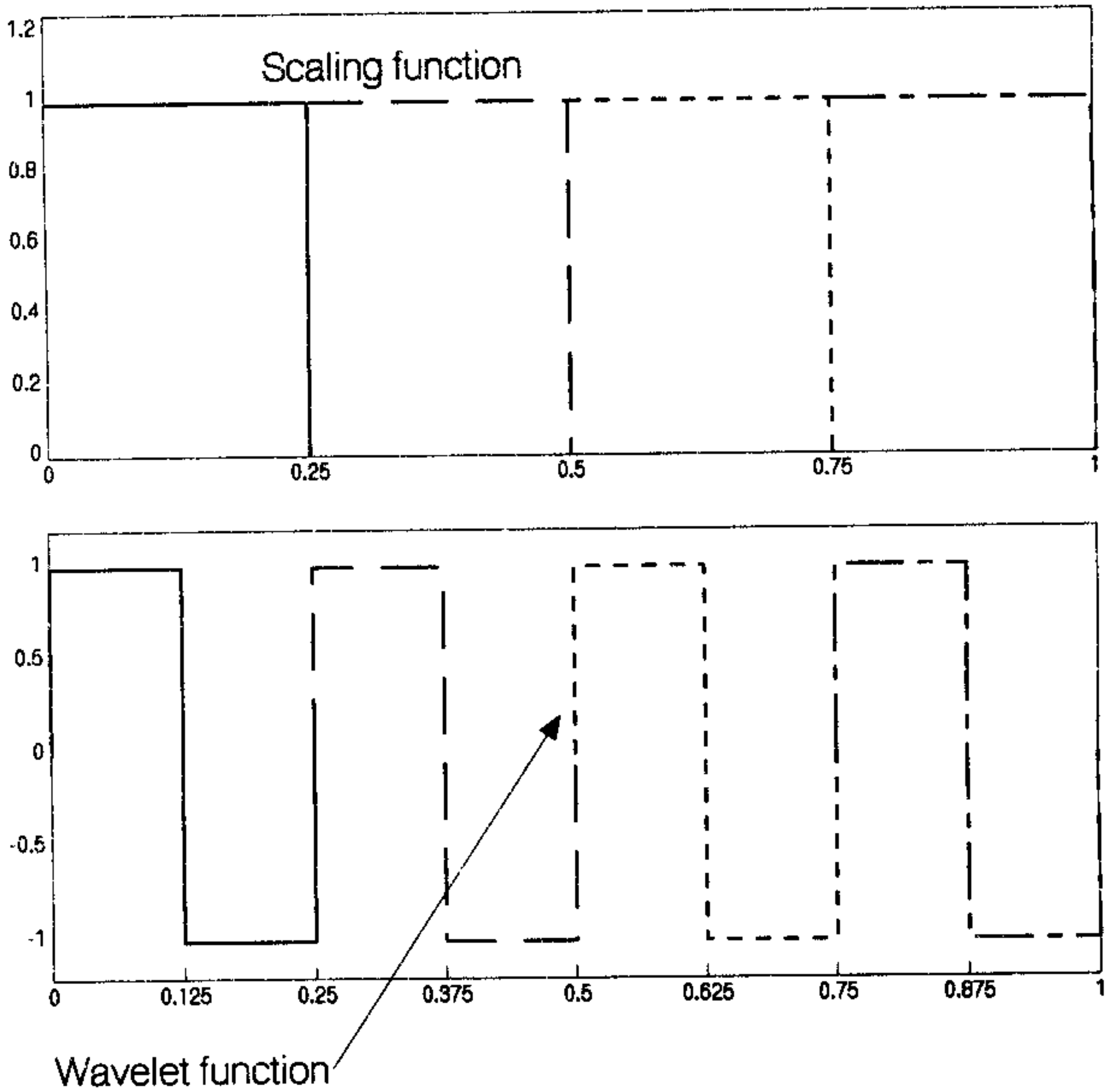

Fig.3 Example construction of a wavelet

greater than or equal to 5. The ratio (25/37)of compaction is 0.68. Also, 25 units of input data and 11 units of transformed data is greater than or equal to 10. The ratio (11/25)of compaction is 0.44. Further it is possible to set to zero those elements which are small in magnitude(thresholding). The inverse transformation then recovers the original matrix without much distortion. It should be noted that the first case presented in the example has threshold equal to zero, and there is no "Thresholded transform data". Considering the fact that rainfall data are random, the accuracy of reconstruction is remarkable as seen in Figs. 4, 5 and 6. Figs.4,5 and 6 have varying values of threshold equal to 2,4 and 8 respectively. The corresponding "Thresholded transform data" shows an increase in the number of zeros. The wavelet explained in this section, was historically the first wavelet, is very simple and suitable for explaining the theory; and even better results can be obtained by using better wavelets.

This wavelet is applied to a string of rainfall data (which is random) in the Ganges catchment. Though the wavelet is simplest the results are very encouraging. Below is presented the original data, thresholded data with the reconstructed data:

Input data -- 64 values

1.9	0	0	18.4	3.5	7.3	0	0.2
2.4	2.3	8.4	3.2	5.0	55.9	10.0	2.3
21.8	8.1	22.2	27.5	2.3	13.2	10.5	24.4
32.5	16.4	0.4	5.0	2.5	33.4	12.0	.2
60.8	10.3	0	1.3	7.4	1.0	6.6	11.7
31.2	36.7	12.5	1.9	9.2	7.1	10.6	7.2
4.3	1.4	2.7	3.4	0	4.1	0.6	32.9
65.0	58.3	29.9	24.6	16.7	5.8	0.6	11.3

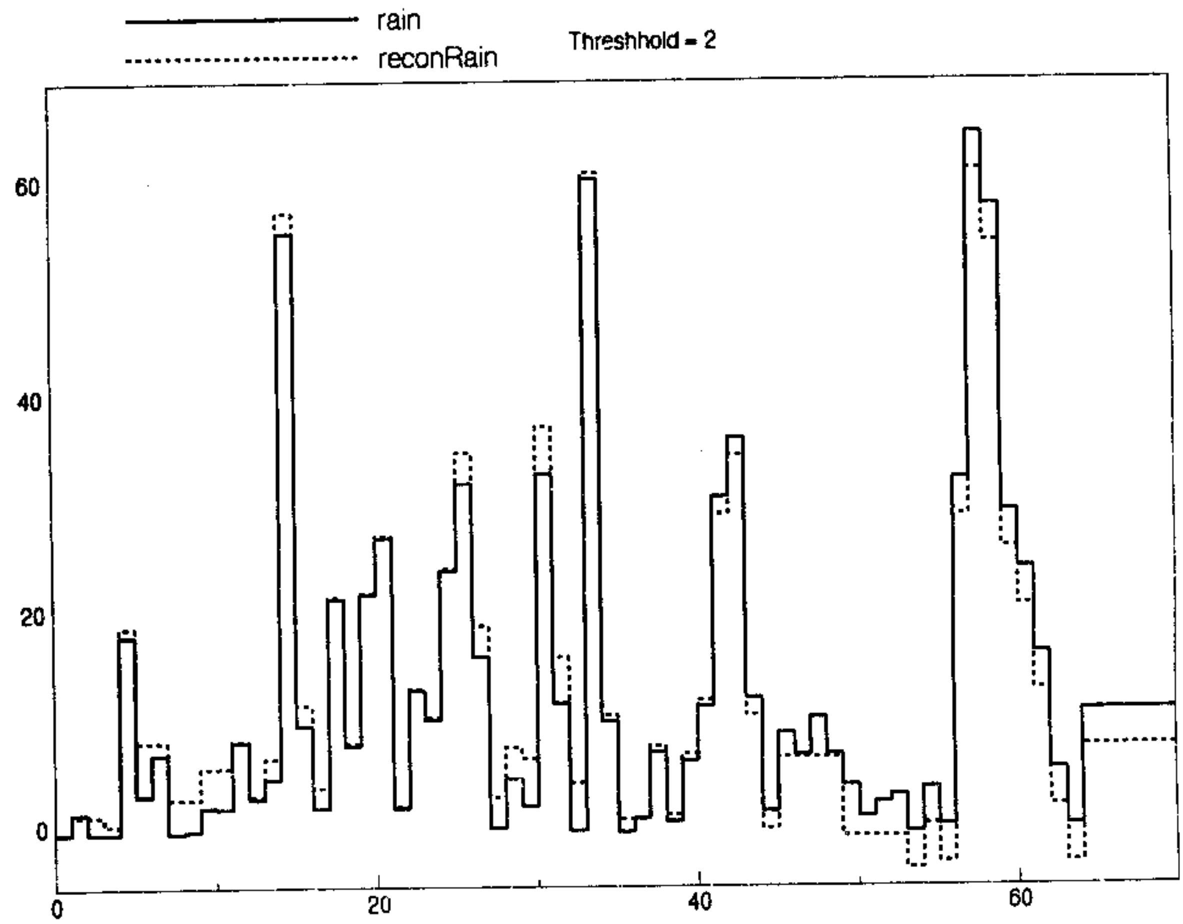

Fig.4 Rainfall data with reconstruction by wavelets

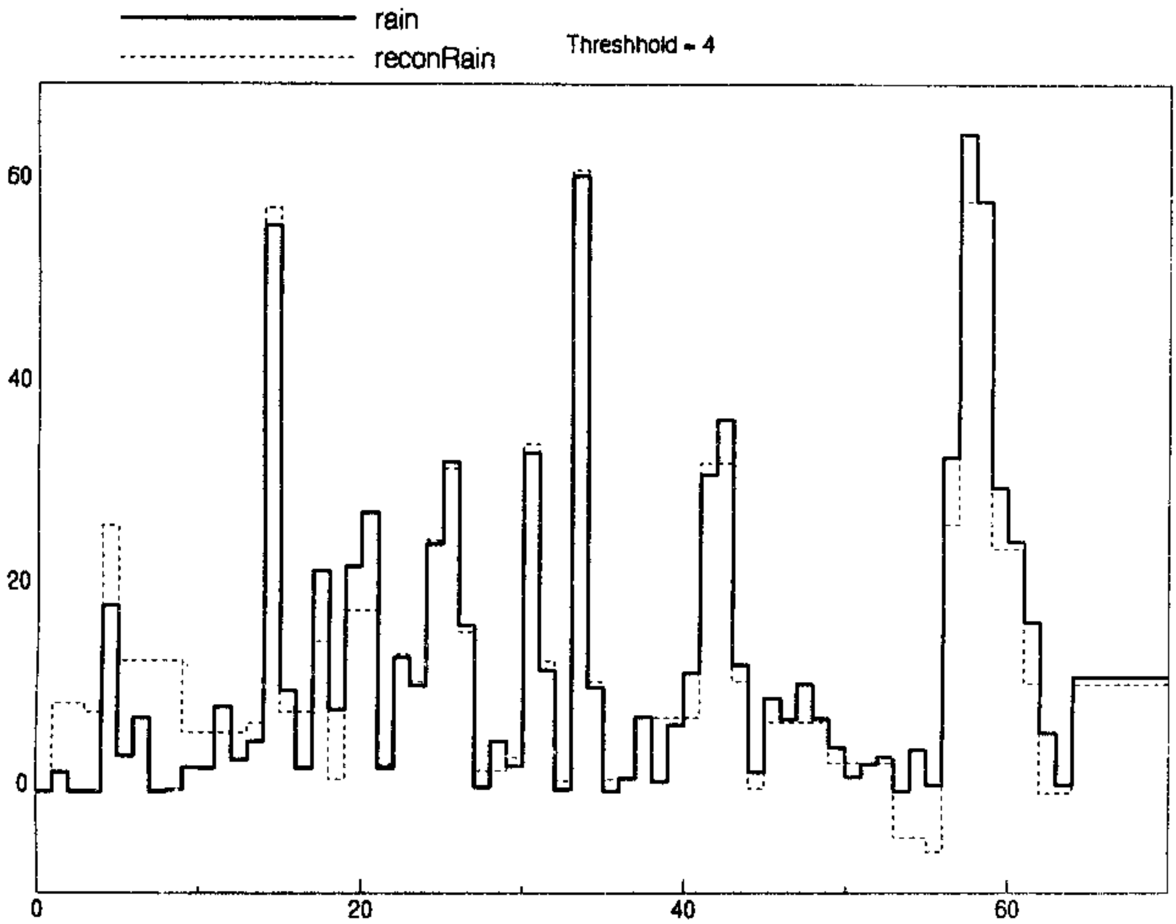

Fig.5 rainfall data with reconstruction by wavelets

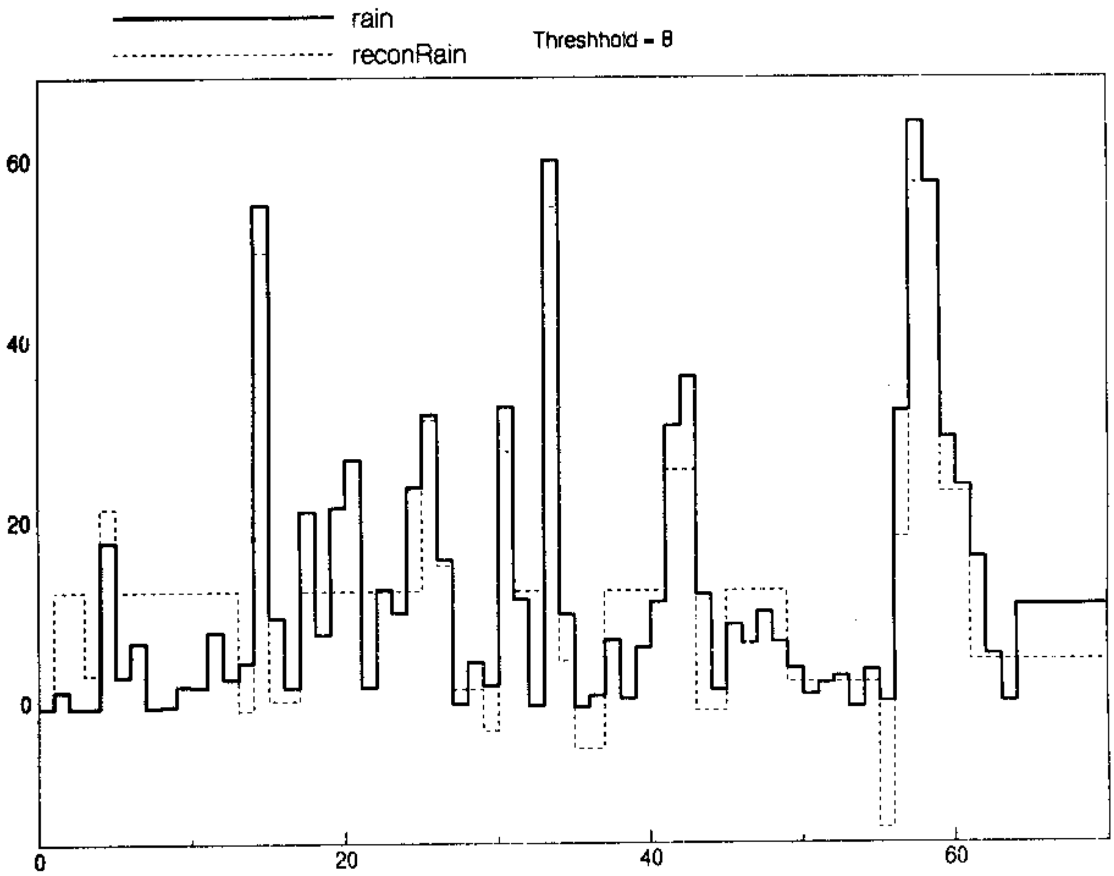

Fig.6 Rainfall data with reconstruction by wavelets

Threshold = 0

Transformed data

12.973	-1.936	-3.488	-1.441	-3.638	1.725	-1.081	-10.175
1.162	-7.113	3.650	.775	5.713	6.025	-3.225	17.925
-4.125	2.650	-1.725	12.150	-4.950	-4.850	10.875	5.925
17.450	-2.475	13.375	-0.375	-0.1	-7.350	17.2	2.650
0.950	-9.2	-1.9	-0.1	0.050	2.6	-25.450	3.850
6.850	-2.650	-5.450	-6.950	8.050	-2.3	-15.450	5.9
25.250	-0.650	3.2	-2.550	-2.750	5.3	1.050	1.7
1.450	-0.350	-2.050	-16.150	3.350	2.650	5.450	-5.350

Reconstructed data

1.9	0	0	18.4	3.5	7.3	0	0.2
2.4	2.3	8.4	3.2	5.0	55.9	10.0	2.3
21.8	8.1	22.2	27.5	2.3	13.2	10.5	24.4
32.5	16.4	0.4	5.0	2.5	33.4	12.0	0.2
60.8	10.3	0	1.3	7.4	1.0	6.6	11.7
31.2	36.7	12.5	1.9	9.2	7.1	10.6	7.2
4.3	1.4	2.7	3.4	0	4.1	0.6	32.9
65.0	58.3	29.9	24.6	16.7	5.8	0.6	11.3

Threshold = 2.

Transformed data

12.973	-1.936	-3.488	-1.441	-3.638	1.725	-1.081	-10.175
1.162	-7.113	3.650	0.775	5.713	6.025	-3.225	17.925
-4.125	2.650	-1.725	12.150	-4.950	-4.850	10.875	5.925
17.450	-2.475	13.375	-0.375	-0.1	-7.350	17.2	2.650
0.950	-9.2	-1.9	-0.1	0.050	2.6	-25.450	3.850
6.850	-2.650	-5.450	-6.950	8.050	-2.3	-15.450	5.9
25.250	-0.650	3.2	-2.550	-2.750	5.3	1.050	1.7

1.450	-0.350	-2.050	-16.150	3.350	2.650	5.450	-5.350

Thresholded transform data

12.973	0	-3.488	0	-3.638	0	0	-10.175
0	-7.113	3.650	0	5.713	6.025	-3.225	17.925
-4.125	2.650	0	12.150	-4.950	-4.850	10.875	5.925
17.450	-2.475	13.375	0	0	-7.350	17.2	2.650
0	-9.2	0	0	0	2.6	-25.450	3.850
6.850	-2.650	-5.450	-6.950	8.050	-2.3	-15.450	5.9
25.250	0	3.2	-2.550	-2.750	5.3	0	0
0	0	-2.050	-16.150	3.350	2.650	5.450	-5.350

Reconstructed data

1.723	1.723	0.773	19.173	8.498	8.498	3.198	3.198
6.011	6.011	8.611	3.411	6.936	57.836	11.936	4.236
22.011	8.311	22.411	27.711	2.511	13.411	10.711	24.611
35.386	19.286	3.286	7.886	6.936	37.836	16.436	4.636
61.386	10.886	1.236	1.236	7.986	1.586	7.186	12.286
29.623	35.123	10.923	0.323	6.948	6.948	6.948	6.948
-0.427	-0.427	-0.427	-0.427	-3.377	0.723	-2.777	29.523
61.623	54.923	26.523	21.223	13.323	2.423	-2.777	7.923

Threshold = 4.

Transformed data

12.973	-1.936	-3.488	-1.441	-3.638	1.725	-1.081	-10.175
1.162	-7.113	3.650	0.775	5.713	6.025	-3.225	17.925
-4.125	2.650	-1.725	12.150	-4.950	-4.850	10.875	5.925
17.450	-2.475	13.375	-0.375	-0.1	-7.350	17.2	2.650
0.950	-9.2	-1.9	-0.1	0.050	2.6	-25.450	3.850
6.850	-2.650	-5.450	-6.950	8.050	-2.3	-15.450	5.9

25.250 -0.650 3.2 -2.550 -2.750 5.3 1.050 1.7

1.450 -0.350 -2.050 -16.150 3.350 2.650 5.450 -5.350

Thresholded transform data

12.973 0 0 0 0 0 0 -10.175

0 -7.113 0 0 5.713 6.025 0 17.925

-4.125 0 0 12.150 -4.950 -4.850 10.875 5.925

17.450 0 13.375 0 0 -7.350 17.2 0

0 -9.2 0 0 0 0 -25.450 0

6.850 0 -5.450 -6.950 8.050 0 -15.450 5.9

25.250 0 0 0 0 5.3 0 0

0 0 0 -16.150 0 0 5.450 -5.350

Reconstructed data

8.848 8.848 7.898 26.298 12.973 12.973 12.973 12.973

5.861 5.861 5.861 5.861 6.786 57.686 7.936 7.936

14.873 1.173 17.923 17.923 2.673 13.573 10.873 24.773

31.898 15.798 2.098 2.098 3.448 34.348 12.948 1.148

61.386 10.886 1.236 1.236 7.261 7.261 7.261 7.261

32.373 32.373 10.923 0.323 6.948 6.948 6.948 6.948

2.798 2.798 2.798 2.798 -4.552 -4.552 -6.2 26.298

58.273 58.273 23.873 23.873 10.673 -0.227 -0.127 10.573

Threshold = 8.

Transformed data

12.973 -1.936 -3.488 -1.441 -3.638 1.725 -1.081 -10.175

1.162 -7.113 3.650 0.775 5.713 6.025 -3.225 17.925

-4.125 2.650 -1.725 12.150 -4.950 -4.850 10.875 5.925

17.450 -2.475 13.375 -0.375 -0.1 -7.350 17.2 2.650

0.950	-9.2	-1.9	-0.1	0.050	2.6	-25.450	3.850
6.850	-2.650	-5.450	-6.950	8.050	-2.3	-15.450	5.9
25.250	-0.650	3.2	-2.550	-2.750	5.3	1.050	1.7
1.450	-0.350	-2.050	-16.150	3.350	2.650	5.450	-5.350

Thresholded transform data

12.973	0	0	0	0	0	0	-10.175
0	0	0	0	0	0	0	17.925
0	0	0	12.150	0	0	10.875	0
17.450	0	13.375	0	0	0	17.2	0
0	-9.2	0	0	0	0	-25.450	0
0	0	0	0	8.050	0	-15.450	0
25.250	0	0	0	0	0	0	0
0	0	0	-16.150	0	0	0	0

Reconstructed data

12.973	12.973	3.773	22.173	12.973	12.973	12.973	12.973
12.973	12.973	12.973	12.973	-0.327	50.573	0.823	0.823
12.973	12.973	12.973	12.973	12.973	12.973	12.973	12.973
31.898	15.798	2.098	2.098	-2.477	28.423	12.973	12.973
55.673	5.173	-4.477	-4.477	12.973	12.973	12.973	12.973
26.348	26.348	-0.402	-0.402	12.973	12.973	12.973	12.973
2.798	2.798	2.798	2.798	2.798	2.798	-13.352	18.948
58.273	58.273	23.873	23.873	5.223	5.223	5.223	5.223

STEPS FOR APPLICATION OF WAVELET TRANSFORMS TO CANAL BREACH

The assumptions of physical model, capability of computational algorithm and knowledge of experimental results are inter-dependent in a complex simulation.

Model:

The two equations which govern the incompressible flow phenomena are:

a)Continuity equation

$$\frac{\partial U_i}{\partial X_i} = 0 \tag{22}$$

b)Navier Stokes equation

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = \frac{-1}{\rho}\frac{\partial P^*}{\partial x_i} + \frac{\partial^2 U_i}{\partial x_j \partial x_j}\nu \tag{23}$$

where U_i are the components of velocity, P^* is sum of pressure, a surface force and γh the gravity body force. Einstein's convention of summation is used, which is terms of repeating subscripts are summed.

The saint venant equations, the depth averaged (using shallow water theory), describing the two dimensional unsteady flows may be written in vector form as

$$\mathbf{H_t + U_x + V_y + S = 0} \tag{24}$$

$$\text{where, } H = \begin{pmatrix} h \\ uh \\ vh \end{pmatrix}; U = \begin{pmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \\ uvh \end{pmatrix}; V = \begin{pmatrix} vh \\ uvh \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} S = \begin{pmatrix} 0 \\ -gh(S_{0x} - S_{fx}) \\ -gh(S_{0y} - S_{fy}) \end{pmatrix} \tag{25}$$

$$\text{and } S_{fx} = \frac{n^2 u \sqrt{u^2+v^2}}{C_0^2 h^{4/3}} \qquad S_{fy} = \frac{n^2 v \sqrt{u^2+v^2}}{C_0^2 h^{4/3}} \tag{26}$$

Application of wavelet element method:

The construction of multi-resolution systems and wavelets on general domains in $\mathbf{R}^N$ is a crucial issue for applying wavelet methods to the numerical solution of operator equations such as partial differential and integral equations.

Tensor products of scaling functions and wavelets are mapped to the sub-domains in which the original domain is split. By matching these functions across the inter-element boundaries, globally continuous bi-orthogonal wavelet systems are obtained, which allow the characterization of certain function spaces and their duals. The coefficient matricies obtained will be sparse.

Boundary conditions:

For free surface flows considering a steady flow might not lead to much simplification. For steady flow, the equations are elliptic and boundary conditions must be specified on all boundaries. Typical values of boundary conditions specified are the values of the variables or its gradients. Known boundary conditions mean that the variables are known apriori. Velocity at the boundary is either to be determined form experiment or from apriori expected behaviour -- such as critical flow occurs at the breach or that no flow can take place across a solid boundary. But the exact location of the critical section poses a problem and it needs to be investigated if a work around can be found.

The gradient boundary condition for the case of free surface flows using Navier Stokes equation are:

$$\text{normal:} \quad -\frac{P + \gamma h}{\rho} + 2\nu \frac{\partial U_n}{\partial x_n} \tag{27}$$

$$\text{traction:} \quad \nu\left(\frac{\partial U_n}{\partial x_t} + \frac{\partial U_t}{\partial x_n}\right) \tag{28}$$

in which n indicates normal to the boundary, t indicates tangential to the boundary and ν is the kinematic viscocity

Commonly, to avoid the complex problem of specifying boundary conditions, governing equations are integrated across the depth and "depth averaged" equations are considered. The assumption is that the length of horizontal dimension, say length of wave, is much greater compared to depth -- hence interestingly many commonly encountered non-shallow wave situations, like a deep or narrow canal, may sometimes give misleading solutions with the application of this method.

In hyperbolic systems, like the Saint Venant equations above, errors present in the specification of the boundary conditions are propagated throughout the domain. The cause of many an instability in computation lies here. Initial conditions of all the independent variables are specified throughout the domain. For a solid boundary the no-slip condition $v/u = \tan\theta$ where θ is the angle between wall and the x-axis.

SCOPE OF APPLICATIONS IN HYDROLOGY

Wavelet transforms can have very wide applications in hydrology. The data compaction needs of the hydrological applications us expected to have great promise. The classical example of data storage is that of the FBI which had to use wavelet methods to store the data of finger prints. A few parameters are sufficient to describe a curve.

The literature and research on wavelets are growing by leaps and bounds and it does promise to have wider applications in hydrology in future.

CONCLUSIONS:

1) The key concepts of wavelet transforms is presented. Information of a function could equivalently be specified in the original domain or in the frequency domain. There is a one to one relationship in the Fourier Transforms between these two domain.
2) Fourier transforms are in capable of analysing data with non-stationary (data whose frequency component changes with time) components.
3) Wavelet and windowed Fourier transforms introduce another dimension into the understanding of the data. This is comparable to catching the tone or shrillness of a voice apart from capturing the intensity. The necessity of the method of analysis as compared to the conventional Fourier analysis becomes inescapable in real time understanding of data. It should be noticed that Fourier analysis can be thought of as post mortem analysis of data --as one has to wait till the data stream ends(meaning of integration between $-\infty$ to $+\infty$).
4) Windowed Fourier analysis is a fixed resolution analysis as compared to Wavelet analysis, which is a multi-resolution analysis. Low and high frequency components of a data set is rationally analysed differently.
5) The mathematical concepts of dual and bi-orthogonal basis which forms the backbone of the wavelet theory is derived from the fundamentals. It should be noticed that how compact the statement appear in the new frame work. Compactness of the derivations are required in reducing unnecessary clutter and to drive home and visualise the meaning of derivations. Dual bases are presented and the important concept of resolution of unity is derived from it. Further, bi-orthogonal bases are presented. Finally, Hilbert spaces are presented which links up the vectors and functions on a firm basis.
6) Singular value decomposition is presented which allows to determine the inverse of all matricies.
7) Very recently Wavelets have been used in the analysis of operator equations. Such a representation allows the solving of equivalent equations where the number of operations are reduced. Wavelets sense the structures within that data and functions,

thus representing them in reduced size. Wavelet element method is one of such applications.

8) Concepts of scale, support, translation, uncertainty and resolution used in wavelet transforms are explained. Algorithms of Wavelet analysis and synthesis are presented. The wavelets chosen for each application will be different and it has been proved that some wavelets cannot be expanded in terms of elementary functions.

9) Discreet wavelet transform has been explained. Sampling theorem has been expalined in detail. The mathematics necessary for their comprehension has been given as appendix.

10) Steps for application of wavelet transforms to canal breach has been presented.

11) Wavelet method of analysis offers great promise for its application in hydrology and hydraulics

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Appendix 1

DUAL AND BI-ORTHOGONAL BASES, FOURIER SERIES, TRANSFORM , W FT AND IMPORTANT CONCEPTS OF MATRIX ALGEBRA

For understanding Wavelets in detail , it is necessary to review the fundamentals from a mathematical language so that the concepts and operations are presented in a concise form in terms of set theory. The preliminary background is important because even the first definition of wavelet transform incorporates the concepts of translation, scale etc., and that of multi resolution. It is to be noted that complex sets need to be considered for the development of theory of wavelets. It is because even for analysing real objects complex methods are simpler than real methods -- complex exponential form of Fourier series is simpler than using real form in terms of sines and cosines.

The sets **R** and **C** denote the set of all real and complex numbers. $\mathbf{R}^M$ denotes the set of all ordered M-tuples of numbers which is usually written as column vector. A basis for $\mathbf{C}^M$ is the collection of vectors $\{\mathbf{b}_1,\mathbf{b}_2 \ldots\ldots\ldots\ \mathbf{b}_M\}$ which are linearly independent and any vector in $\mathbf{C}^M$ can be written as a linear combination of them. If $\mathbf{u} \in \mathbf{C}^M$ then $\mathbf{u}^m$ is the m^{th} component of $\mathbf{u}$.

Given $M,N \in \mathbf{N}$ a function F from $\mathbf{C}^M$ to $\mathbf{C}^N$ is defined by specifying a rule that assigns a vector $F(\mathbf{u}) \in \mathbf{C}^N$ to each vector $\mathbf{u} \in \mathbf{C}^M$. The set of all such functions is denoted by $L(\mathbf{C}^M , \mathbf{C}^N)$. $L(\mathbf{C}^M , \mathbf{C}^N)$ is also denoted by $F \in \mathbf{C}_M^N$ and also by $F:\mathbf{C}^M \rightarrow \mathbf{C}^N$. By such a notation linear operators are identified with matricies. The basis vectors $\{\mathbf{b}_1,\mathbf{b}_2 \ldots\ldots\ldots\ \mathbf{b}_N\}$span $\mathbf{C}_1^N = \mathbf{C}^N$.

The set $\mathbf{C}_N$ consists of row vectors with N complex entries. The corresponding basis $\{B^1, B^2 \ldots B^N\}$ called the 'dual basis' of $\{\mathbf{b}_1,\mathbf{b}_2 \ldots\ldots\ldots\ \mathbf{b}_N\}$, is constructed below. The expression $\mathbf{u} = \sum_{n=1}^{N} \mathbf{b}_n u^n$ specifies a unique vector $\mathbf{u} \in \mathbf{C}^N$ for any basis $\{\mathbf{b}_1,\mathbf{b}_2 \ldots\ldots\ldots$ $\mathbf{b}_N\}$, which is not necessarily orthogonal. For each n $B^n(\mathbf{u}_e)$ is defined to be equal to the

n^{th} component of **u**, i.e, $B^n(\mathbf{u}_e) \equiv u^n$, where $\mathbf{u}_e$ is the vector **u** expanded with respect to the standard basis.

The vectors basis $\{B^1, B^2 ... B^N\}$ spans C_N:

Proof: Let F_n, scalar coefficients be defined by $F(\mathbf{b}_n) \in \mathbf{C}$

$$\mathbf{Fu} = F(\mathbf{u} = \sum_{n=1}^{N} \mathbf{b}_n u^n) = \sum_{n=1}^{N} F(\mathbf{b}_n) u^n = \sum_{n=1}^{N} F_n u^n = (\sum_{n=1}^{N} F_n B^n)\mathbf{u} \quad (29)$$

hence,

$$F = \sum_{n=1}^{N} F_n B^n \quad (30)$$

An example of resolution of unity or identity in terms of pair of dual basis is:

$$\sum_{n=1}^{N} \mathbf{b}_n B^n = I \quad (31)$$

The operator $\mathbf{b}_n B^n$ projects any vector to its vector component parallel to $\mathbf{b}_n$. The sum of these projections give the complete vector.

If $\{\mathbf{d}_n\}$ be another basis in $\mathbf{C}^N$ and if $\{D^n\}$ be its dual basis,

$$\sum_{n=1}^{N} \mathbf{d}_n D^n = I \quad (32)$$

then,

$$u^n = B^n\mathbf{u} = B^n I\mathbf{u} = B^n \sum_{k=1}^{N} \mathbf{d}_k D^k \,\mathbf{u} = B^n \sum_{k=1}^{N} \mathbf{d}_k w^k = \sum_{k=1}^{N} (B^n \mathbf{d}_k) w^k = \sum_{k=1}^{N} T_k^n w^k \quad (33)$$

'T' therefore is a NxN matrix of transformation.

Inner products, adjoint operator :

The standard inner product in $\mathbf{C}^N$ is defined as

$$\langle \mathbf{u},\mathbf{v} \rangle = \sum_{n=1}^{N} \bar{u}^n v^n, \ \mathbf{u},\mathbf{v} \in \mathbf{C}^N \quad (34)$$

where, u^n and v^n are the component of **u** and **v** w.r.t. standard basis. $\bar{u}$ denotes the complex conjugate of u. The norm of vector **u** is defined to be $||\mathbf{u}|| = <\mathbf{u},\mathbf{u}> = \sum_{n=1}^{N} |u^n|^2$.

Any inner product should satisfy the following fundamental properties

(P) Positivity: $||\mathbf{u}|| > 0$ for all $\mathbf{u} > 0$ (35)

(H) Hermiticity : $\overline{<\mathbf{u},\mathbf{v}>} = <\mathbf{v},\mathbf{u}>$ (36)

(L)Linearity: $<\mathbf{u},c\mathbf{v}+\mathbf{w}> = c<\mathbf{u},\mathbf{v}>+<\mathbf{u},\mathbf{w}>$ (37)

The physics convention (opposite to the mathematical convention) chosen thus makes the scalar product linear in the second factor. It is anti-linear in the first factor

$$<c\mathbf{u}+\mathbf{w}, \mathbf{v}> = \bar{c}<\mathbf{u},\mathbf{v}>+<\mathbf{w},\mathbf{v}> \quad (38)$$

Another example of inner product is the weighted inner product

$$<\mathbf{u},\mathbf{v}> = \sum_{n=1}^{N} \mu_n \bar{u}^n v^n \quad \text{where } \mu_n > 0 \quad (39)$$

Hence the concept of orthogonality and length are dependent on the choice of the inner product.

Adjoints are a powerful concept in linear algebra. The concept extended to functional spaces is as follows: The adjoint of an operator $F \in \mathbf{C}_M^N$ is an operator $F^* \in \mathbf{C}_N^M$ such that

$$<\mathbf{u}, F^* \mathbf{v}> = <F\mathbf{u},\mathbf{v}> \text{ for all } \mathbf{u} \in \mathbf{C}^M \text{ and } \mathbf{v} \in \mathbf{C}^N \quad (40)$$

If $\{\mathbf{a}_1,\mathbf{a}_2 \ldots\ldots\ldots\ldots \mathbf{a}_M\},\{\mathbf{b}_1,\mathbf{b}_2 \ldots\ldots\ldots\ldots \mathbf{b}_N\}$ are chosen as bases such that they orthonormal with respect to the chosen inner product.

$$F\mathbf{a}_m = \sum_{n=1}^{N} \mathbf{b}_n F_m^n \text{ and } F^* \mathbf{b}_n = \sum_{m=1}^{M} \mathbf{a}_m (F^*)_n^m \quad (41)$$

$$F_m^n = <\mathbf{b}_n, F\mathbf{a}_m> \text{ and } (F^*)_n^m = <\mathbf{a}_m, F^* \mathbf{b}_n> \quad (42)$$

$$(F^*)_n^m = <F\mathbf{a}_m, \mathbf{b}_n> = \overline{<\mathbf{b}_n, F\mathbf{a}_m>} = \bar{F}_m^n \quad (43)$$

The matrix F^* is the Hermitian conjugate the matrix F.

Representation of inner product as an actual product:

On the one dimensional space **C**, choosing standard inner product

$$<c,c'> = \bar{c}\,c' \tag{44}$$

and regarding **u** as an operator $\mathbf{u} \in \mathbf{C}^N$ defined by **uc=cu** then the adjoint operator $\mathbf{u}^* \in \mathbf{C}_N$ determined below:

By the definition of the inner product (10)

$$< c,\mathbf{u}^*\mathbf{v} >= \bar{c}\,\mathbf{u}^*\mathbf{v} \tag{45}$$

$$< c,\mathbf{u}^*\mathbf{v} >=< \mathbf{u}c,\mathbf{v} >= \bar{c} < \mathbf{u},\mathbf{v} > \tag{46}$$

From Eqs. 17 and 18

$$<\mathbf{u},\mathbf{v}> = \mathbf{u}^*\mathbf{v} \tag{47}$$

Dirac called $\mathbf{u}^*$ as `bra' **v** as `ket'. The above derivation is basis independent but depends on the choice of inner product.

It can be shown that

$$F^{**} = F \tag{48}$$

$$(GF)^* = F^*G^* \tag{49}$$

Every `linear functional' (Defined to be $F \in \mathbf{C}_N^1$) can be written as $F = \mathbf{u}^*$ for a unique vector $\mathbf{u} \in \mathbf{C}^N$ and $\mathbf{u} = F^*$.

Bi-orthogonality:

Each $B^k \in \mathbf{C}^N$ determines a unique vector $\mathbf{b}^k$ defined to be $(B^k)^* \in \mathbf{C}^N$

$$<\mathbf{b}^k,\mathbf{b}_n>=(\mathbf{b}^k)^*\mathbf{b}_n=B^k\mathbf{b}_n=\delta_n^k \tag{50}$$

since it can be easily shown that

$$B^n\mathbf{b}_k=\delta_k^n \tag{51}$$

The vectors $\{\mathbf{b}^k\}$ form another basis of $\mathbf{C}^N$

The above relation is called the DUALITY RELATION. Eq.42 states that bases $\{\mathbf{b}^n\}$ and $\{\mathbf{b}_n\}$ mutually bi-orthogonal. From the relation $B^* = (B^*)^{**} = (\mathbf{b}^*)^*$ and from Eq.31

$$\sum_{n=1}^{N} \mathbf{b}_n (\mathbf{b}^n)^* = I \tag{52}$$

This gives the expansion of any vector in terms of $\mathbf{b}_n$

$$\mathbf{u} = \sum_{n=1}^{N} \mathbf{b}_n u^n \quad \text{where } u^n = (\mathbf{b}^n)^* \mathbf{u} = \langle \mathbf{b}^n, \mathbf{u} \rangle \tag{53}$$

Taking adjoint of Eq.52,

$$\sum_{n=1}^{N} \mathbf{b}^n \mathbf{b}_n^* = I \tag{54}$$

$$\mathbf{u} = \sum_{n=1}^{N} \mathbf{b}^n u_n \quad \text{where } u_n = (\mathbf{b}_n)^* \mathbf{u} = \langle \mathbf{b}_n, \mathbf{u} \rangle \tag{55}$$

When the given basis is ortho normal with respect to the inner product

$$\sum_{n=1}^{N} \mathbf{b}_n \mathbf{b}_n^* = I \tag{56}$$

with the usual expansion of vectors $u^n = (\mathbf{b}_n)^* \mathbf{u} = \langle \mathbf{b}_n, \mathbf{u} \rangle$

Functional spaces and Hilbert spaces:

A generalisation of the concept of vectors is presented below:

Let $\eta = \{1,2,3......N\}$ be the set of integers and $f:\eta \rightarrow C$. The generalisation is from the fact that the function is now specified which assigns a complex number f(n) to each $n \in \eta$. The set of all such functions is denoted by $\mathbf{C}^\eta$. A one to one correspondence is established between $\mathbf{C}^\eta$ and $\mathbf{C}^N$ by writing f(n) as column vector $\mathbf{u}_f \in \mathbf{C}^N$, such that $(\mathbf{u}_f)^n$ is defined to be f(n). Further let S be an arbitrary set and let $f:S \rightarrow C$. Defining vector operations in $\mathbf{C}^S$ by $(cf)(s) = c\, f(s)$ and $(f+g)(s) = f(s)+g(s)$ for $s \in S$. Then $\mathbf{C}^S$ is a complex vector space. If S is infinite dimensional it becomes a infinite dimensional vector space. For example $\mathbf{C}^R$ is vector space of all possible, including non-linear, functions $f:\mathbf{R} \rightarrow \mathbf{C}$.

Most ideas of linear algebra can be extended to $\mathbf{C}^S$ if suitable restrictions are placed on function f. The inner product and the norm are defined respectively by

$$<f,g> = \sum_{n=-\infty}^{\infty} \overline{f(n)}\, g(n),\ f,g \in \mathbf{C}^{\mathbf{Z}} \qquad (57)$$

$$||f|| = <f,f> = \sum_{n=1}^{N} | f(n) |^2 \qquad (58)$$

One solution to the problem of convergence of series associated with infinite dimensional vectors is to consider only the subset of functions with finite norm.

$\aleph$ is defined to be $\{f \in \mathbf{C}^{\mathbf{Z}}: ||f||^2 < \infty\}$ (59)

$||f+g|| \le ||f|| + ||g|| < \infty$ by triangle inequality (60)

With the inner product as defined $\aleph$ is called the space of square summable complex sequences $\lambda^2(\mathbf{Z})$.

Two problems exist for extending the analysis to infinite dimensions for continuous S with the inner product as defined below:

$$<f,g> = \int_{-\infty}^{\infty} \overline{f(t)}\, g(t) dt,\ f,g \in \mathbf{C}^{\mathbf{R}} \qquad (61)$$

$$||f|| = <f,f> = \int_{-\infty}^{\infty} |f(t)|^2\, dt \qquad (62)$$

1)The Reimann's definition of integral fails

2)The positivity condition fails

The first problem is remedied by recasting the integral in the Lebegue sense, which is based on the concept of measure, which in turn is the generalisation of concept of length. The second problem is solved by generalising the concept of function to include sets of functions and to regard two measurable functions f and g as identical if the set of points on which they differ

$D = \{t \in \mathbf{R}: f(t) \neq g(t)\}$ (63)

has measure zero. $\aleph$ is called the space of square-integrable complex valued functions on R and is denoted by $L^2(\mathbf{R})$.

A Hilbert space is any vector space with the inner product satisfying the conditions of Positivity, Hermiticity and Linearity and moreover complete in the sense that any sequence $\{f_1, f_2 \ldots\}$ in $\aleph$ for which $||f_n - f_m||$ -> zero when m and n both tend to ∞ converges to some $f \in \aleph$ in the sense that $||f-f_n||$ -> zero as n tends to ∞. Examples of Hilbert spaces are $\lambda^2(\mathbf{Z})$ & $L^2(\mathbf{Z})$.

Supported square integrable functions on R:

A function belonging to $L^2[\mathbf{R}]$ is said to be supported in the interval [a,b], supp f $\subset$[a,b], if the set of points outside [a,b] at which f(t) not equal to zero has zero measure. It is denoted by $L^2[a,b]$. It is the subspace of $L^2(\mathbf{R})$ and is itself also a Hilbert space. Given an arbitrary function $f \in L^2(\mathbf{R})$, f has compact support if supp [a,b] for a bounded interval $[a,b] \subset \mathbf{R}$.

Adjoint operator generalised to Hilbert space setting:

Consider two Hilbert spaces $\aleph$ and and a function $A:\aleph \to \Im$, a is linear if

$$A(cf+g) = cA(f)+A(g) \text{ for all } c \in \mathbf{C} \text{ and } f,g \in \aleph \quad (64)$$

A is then called an operator. A function on a infinite dimensional Hilbert space to be a operator requires apart from linearity the requirement that it be bounded

$$||f|| \leq C||f|| \; ; \; C > 0 \text{ for all } f \in \aleph \quad (65)$$

The vector $h \in \aleph$ regarded as a linear map $h:\mathbf{C} \to \aleph$ defines a linear functional on $\aleph$

$h^*:\aleph \to \mathbf{C}$ by h^*g defined to be $\langle h,g \rangle$ for all $g \in \aleph$

h^* is also bounded, since

$$|h^*g| = |\langle h,g \rangle| \leq ||h|| \, ||g|| \quad (66)$$

Let $\aleph$ and $\Im$ be Hilbert spaces and $A: \aleph \to \Im$ be a bounded operator. Then there exists a unique operator $A^*: \Im \to \aleph$ satisfying

$$\langle h,A^*k \rangle = \langle Ah,k \rangle \text{ for all } h \in \aleph \text{ and } k \in \Im \quad (67)$$

Further the adjoint of adjoint is the original operator

$$A^{**} = A \quad (68)$$

and

$$(BA)^* = A^*B^* \quad (69)$$

Mathematical representation of Fourier series and integrals:

The Fourier series is concisely written

$$f = \frac{1}{T}\sum_{n=-\infty}^{\infty} c_n e_n \tag{70}$$

where T is the period $e_n(t) = e^{2\pi int/T}$ are the harmonic nodes. Let g(u) be a function with supp g ⊂ [-T,0] and defining

$$f_t(u) = \bar{g}(u-t)f(u) \tag{71}$$

the windowed Fourier transform is defined to be

$$\bar{f}(w,t) = f_t(w) = \int_{-\infty}^{\infty} e^{-2\pi iwu} f_t(u)du \tag{72}$$

Defining $g_{w,t}(u) = e^{2\pi i\omega u}g(u-t)$

$$\bar{f}(\omega,t) = g_{w,t}^{*} f \tag{73}$$

The corresponding reconstruction formula is

$$f = \frac{1}{\| g \|}\iint g_{w,t}\bar{f}(w,t)dwdt \tag{74}$$

Frames:

Generalised frames is a general method of analysing and reconstructing functions. It is a tool by which many wavelet like analysis can be developed, studied and compared. Theory of frames generalises the concept of resolution of unity. A basis gives rise to resoultion of unity but not every resolution of unity comes from a basis. It is a key idea in the wavelet analysis. If H be a Hilbert space and let M be a measure space with measure μ, a generalised frame in H indexed by M is a family of vectors H_M defined to be that each element of H_M, h_m belongs to H and m belongs to M; such that

a) For every f belonging to H the function $\tilde{f}$ called the transform of f defined by $\tilde{f}(m) = \langle h_m, f \rangle_H$ is measurable.

b) There are a pair of constants $0 < A \le B < \infty$ such that for every f belonging to

H; $A\|f\|_H^2 \le \| \tilde{f} \|_{L^2}^2 \le B\|f\|_H^2$.

Ill-Conditioned Matricies:

When many equations need to be solved simultaneously, the effect of round off may cause large effects on the results. In certain cases as in Fourier transforms the results are particularly sensitive to round off --- such systems are called ill-conditioned. Number of computations needs to be kept to a minimum by various means like L-U decomposition or FFT.

Singular value decomposition(SVD):

A symmetric matrix A can be expressed as $\mathbf{A = UDU^{-1}}$ where D is diagonal and U is orthogonal $\mathbf{U_j}$ Eigen vectors of A. and α_{ij} are Eigen Values of A. Singular Value Decomposition generates this for arbitrary maticies. In other words every matrix can be inverted with SVD.

Let A =[α_j^i] be a NxN matrix , $\lambda_1\lambda_1....\lambda_N$ be its eigen values.

Arranging the eigen vectors in columns in the Matrix $B = [\beta_j^i]$, a NxN matrix,

$$[\alpha_n^i \beta_1^n] = \lambda_1 \beta_1^i,\ [\alpha_n^i \beta_2^n] = \lambda_2 \beta_2^i \text{ and } [\alpha_n^i \beta_3^n] = \lambda_3 \beta_3^i \tag{75}$$

$$[\alpha_n^i \beta_j^n] = \lambda_j \beta_j^i \quad i = 1,2....N \tag{76}$$

or arranging the eigen values in a NxN matrix in a diagonal matrix λ where

$$\lambda_j^i = \lambda_j \delta_j^i \tag{77}$$

$$AB=B\lambda \tag{78}$$

$$B^{-1}\,AB=\lambda \tag{79}$$

Also, if A is symmetric, then the eigen vectors are orthogonal

$$B^{-1}B = I \tag{80}$$

Theorem 1 of SVD:

Given a MxN matrix A there exists an MxN diagonal matrix D together with orthogonal square matricies U and V such that $A = UDV^T$

If r= rank of A Matrix D can be so arranged so that the only non-zero entries of D are the positive square roots $d_1, d_2 \ldots d_r$ of the non-zero eigen values of AA^T(known as singular values of A) listed in non-decreasing order down the principal diagonal of D.

The columns of U(reciprocal of V) form a full set of eigen vectors of AA^T.

The above theorem is true for all kinds of matricies A.

Definition of pseudo inverses:

If $D = [d_n^m]$ is a MxN diagonal matrix, the elements of pseudo inverse $D^s = [\Delta_m^n]$ of D has the shape NxM, and its elements are given by $\Delta_m^n = 1/d_n^m$

Theorem 2 of SVD: Every matrix is invertible

If $A = UDV^T$ is a singular value decomposition then $A^s = VD^s U^T$. In the special case when A is a square matrix and invertible then the pseudo inverse A^s reduces to the ordinary inverse A^{-1}

The corollary of theorem 2 is that every set of equations are solvable, since an inverse always exists.

Detailed concepts and relationships of Fourier series and transforms:

Representation of function as a sum of Even and odd functions

Any function can be represented as a sum of even and odd function

$f(x) = E(x)+O(x)$ where $E(x) = E(-x)$ and $O(x) = -O(-x)$ (81)

Proof:

$f(-x) = E(x) - O(-x)$ (82)

Solving for E(x) and O(x)

$E(x) = (1/2)(f(x) + f(-x))$ (83)

and

$O(x) = (1/2)(f(x) - f(-x))$ (84)

Convolution:

Convolution is defined as

$$h(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du = f(x)*g(x) \quad (85)$$

This particular form of h(x) has the important property of x-shift invariance. If, $h(x) = f(x)*g(x)$ then,

$f(x-a) * g(x) = h(x-a)$ (86)

Denoting by bars the Fourier transforms:

$\overline{f*g} = \overline{f}\ \ \overline{g}$ (87)

$\overline{fg} = \overline{f} * \overline{g}$ (88)

$f*g = g*f$ (Commutative property) (89)

$f*(g*h) = (f*g)*h$ (Associative property) (90)

$f*(g+h)=f*g+f*h$ (Distributive property) (91)

$\overline{f*g*h} = \overline{f}\ \ \overline{g}\ \ \overline{h}$ (92)

$$\overline{f*(gh)} = \bar{f} \quad (\bar{g} \quad * \quad \bar{h}) \tag{93}$$

$$\int_{-\infty}^{\infty}(f*g)dx = \int_{-\infty}^{\infty}f(x)dx + \int_{-\infty}^{\infty}g(x)dx \tag{94}$$

Parseval's theorem:

If p(x) is real and periodic with period T

$$\frac{1}{T}\int_{-T/2}^{T/2}[p(x)]^2 dx = a_0{}^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2 + b_n{}^2) \tag{95}$$

where the a's and b's are the Fourier series coefficients for p(x)

Serial product:

It is known that a polynomial A ($\Sigma a_i x^i$) multiplied by polynomial B($\Sigma b_i x^i$) results in a polynomial C($\Sigma c_i x^i$) where, $c_k = \sum_{i=0}^{k} a_i b_{k-i}$; k = 0,1,2….. Now the coefficients could be written in terms just its coefficients in their proper place. It is noticed that convolution of two functions (Function g(x) is arranged in reverse order corresponding to g(x-u) which, indeed is plotted reverse in g(u) vs. u plane with line of symmetry at x/2)

$$h(x) \quad = \quad \int_{-\infty}^{\infty} f(u)g(x-u)du \quad = f(x)*g(x) \tag{96}$$

in terms of their discrete values is very similar to the above process of serial product of polynomials. If two functions f(x) and g(x) are represented by:

$$\{f_0 \; f_1 \; f_2 \ldots\ldots\} \tag{97}$$

$$\{g_0 \; g_1 \; g_2 \ldots\ldots\} \tag{98}$$

the serial product, $\{h_0 \; h_1 \ldots\ldots\}$ is defined to be

$$\{f_0 g_0 \quad f_0 g_1 + f_1 g_0 \quad f_0 g_2 + f_1 g_1 + f_2 g_0 \ldots\ldots\} \tag{99}$$

The i[th] term is

$$\sum_{i=0}^{k} f_i g_{k-i} \,. \tag{100}$$

In terms of Matrix notation:

$$\{f_0\ f_1\ f_2 \ldots\ldots f_m\} \tag{101}$$

$$\{g_0\ g_1\ g_2 \ldots\ldots g_n\} \tag{102}$$

$$\{h_0\ h_1 \ldots\ldots h_{m+n}\} \tag{103}$$

$$\begin{bmatrix} h_0 \\ h_1 \\ \cdot \\ \vdots \\ \ddots \\ \cdot \\ h_{m+n} \end{bmatrix} = \begin{bmatrix} f_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_1 & f_0 & 0 & 0 & 0 & 0 & 0 \\ f_2 & f_1 & f_0 & 0 & 0 & 0 & 0 \\ \cdot & f_2 & f_1 & f_0 & 0 & 0 & 0 \\ f_m & \cdot & f_2 & f_1 & f_0 & 0 & 0 \\ 0 & f_m & \vdots & f_2 & f_1 & f_0 & 0 \\ 0 & 0 & f_m & \cdot & f_2 & f_1 & f_0 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ \cdot \\ \vdots \\ g_n \\ 0 \\ 0 \end{bmatrix} \tag{104}$$

Special Functions:

Rectangle function of unit area

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1/2 & |x| = 1/2 \\ 1 & |x| < 1/2 \end{cases} \tag{105}$$

Displaced rectangle function of height h, area hb and centered at c

$$h\Pi(\frac{x-c}{b}) = \begin{cases} 0 & |x-c| > b/2 \\ 1/2 & |x-c| = b/2 \\ 1 & |x-c| < 1/2 \end{cases} \tag{106}$$

$\tau^{-1}\Pi(x/\tau)$ is a rectangle function of height τ^{-1} and base τ of unit area

Triangular Function:

$$\Lambda(x) = \begin{cases} 0 & |x| > 1 \\ 1-|x| & |x| < 1 \end{cases} \tag{107}$$

Heaviside's unit step function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases} \tag{108}$$

$$\Pi(x) = H(x + 1/2) - H(x - 1/2) \tag{109}$$

Ramp function:

$$R(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases} \tag{110}$$

$$R(x) = xH(x) \tag{111}$$

$$R(x) = \int_{-\infty}^{x} H(u)du \tag{112}$$

$$R'(x) = H(x) \tag{113}$$

Convolution with H(x) means integration:

$$H(x)*f(x) = \int_{-\infty}^{\infty} f(u)H(x-u)du \quad = \quad \int_{-\infty}^{x} f(u)du \tag{114}$$

Or,

$$f(x) = \frac{d}{dx}(H(x) * f(x)) \tag{115}$$

$$\operatorname{sgn} x = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} \tag{116}$$

$$\operatorname{sgn} x = 2H(x) - 1 \tag{117}$$

Sinc x

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x} \tag{118}$$

$$\operatorname{sinc} 0 = 1 \tag{119}$$

$\text{sinc } n = 0 \; n = \text{non-zero integer}$ (120)

$$\int_{-\infty}^{\infty} \text{sinc } x \, dx = 1 \qquad (121)$$

$$\int_{-\infty}^{\infty} \text{sinc } ax \, dx = \frac{1}{|a|} \qquad (122)$$

$$Si\ x = \int_{0}^{x} \frac{\sin u}{u} du \qquad (123)$$

$$\frac{Si\ \pi x}{\pi} = \int_{0}^{x} \frac{\sin u}{u} du \qquad (124)$$

$$\text{sinc } x = \frac{d}{dx} \frac{Si\ \pi x}{\pi} \qquad (125)$$

The integral of sinc x

$$H(x) * \text{sinc } x = \int_{-\infty}^{x} \text{sinc } u \, du = \frac{1}{2} + \frac{Si(\pi x)}{\pi} \qquad (126)$$

Square of Sinc x

$$\text{sinc}^2 x = \left(\frac{\sin \pi x}{\pi x} \right)^2 \qquad (127)$$

$\text{sinc}^2\ 0 = 1$ (128)

$\text{sinc}^2\ n = 0 \; n = \text{non-zero integer}$

$$\int_{-\infty}^{\infty} \text{sinc}^2 x \, dx = 1 \qquad (129)$$

Impulse function:

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1 \qquad (130)$$

where, $\delta(x) = 0,\ x \neq 0$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = \underset{\tau \to 0}{Lt} \int_{-\infty}^{\infty} \tau^{-1} \Pi(\frac{x}{\tau}) dx \tag{131}$$

Sifting property is

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0) \tag{132}$$

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a) \tag{133}$$

$$\int_{-\infty}^{\infty} \delta(x) f(x-a)\, dx = f(-a) \tag{134}$$

$$\delta(x) * f(x) = f(x) * \delta(x) = \frac{f(x+) + f(x-)}{2} \tag{135}$$

$$f(x)\delta(x) = f(0) * \delta(x) \tag{136}$$

$$\int_{-\infty}^{\infty} \delta(x) x\, dx = 0 \tag{137}$$

$$x\delta(x) \equiv 0 \tag{138}$$

Though

$$\begin{aligned} &\underset{\tau \to 0}{Lt} [x\tau^{-1}\Pi(\frac{x}{\tau})dx] = 0 \quad \text{for all x;} \\ &\underset{\tau \to 0}{Lt} [x\tau^{-1}\Pi(\frac{x}{\tau})dx]_{max} = \frac{1}{2} \end{aligned} \tag{139}$$

and the limit of the minimum value = -1/2

Null function:

$$\delta^0(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases} \tag{140}$$

$$\int_{-\infty}^{\infty} | \delta^0(x) |\, dx = 0 \tag{141}$$

$$\delta^0(i-j) = \delta_{ij}(x) = \begin{cases} 0 & x \neq 0 \\ 1 & i = j \end{cases} \quad \text{Kroneker delta} \tag{142}$$

$$111(x) = \sum_{n=-\infty}^{\infty} \delta(x-n) \tag{143}$$

$$111(x)=0 \quad x \neq n \tag{144}$$

Periodic sampling property:

$$111(x) f(x) = \sum_{n=-\infty}^{\infty} f(n)\delta(x-n) \tag{145}$$

Replicating property:

The function f(x) appears in replica at unit intervals along the entire length of x axis

$$111(x) * f(x) = \sum_{n=-\infty}^{\infty} f(x-n) \tag{146}$$

$111(x) \supset 111(s)$; $\supset$ refers to the Fourier transform pair

$$111(ax) = \frac{1}{|a|} \sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{a}) \tag{147}$$

$$111(-x) = 111(x) \tag{148}$$

$$111(x+n) = 111(x) \text{ n integral} \tag{149}$$

$$111((x-1/2) = 111(x+1/2) \tag{150}$$

$$\int_{n-1/2}^{n+1/2} 111(x)\, dx = 1 \tag{151}$$

$$11(x) = \frac{1}{2}\delta(x+\frac{1}{2}) + \frac{1}{2}\delta(x-\frac{1}{2}) \tag{152}$$

$$1_1(x) = \frac{1}{2}\delta(x+\frac{1}{2}) - \frac{1}{2}\delta(x-\frac{1}{2}) \tag{153}$$

Useful Fourier transforms:

Fourier Transforms exhibiting symmetric properties

$$\begin{cases} F(s) & = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs} dx \\ f(x) & = or \int_{-\infty}^{\infty} F(s) e^{+i2\pi xs} ds \end{cases} \qquad f(x) \supset F(s) \tag{154}$$

$$F(s) = \int_{-\infty}^{\infty} \text{sinc}\, x\, e^{-i2\pi sx} dx = \Pi(s) \tag{155}$$

$$\text{sinc x} \supset \Pi(s) \tag{156}$$

$$\text{sinc}^2 \text{ x} \supset \Lambda(s) \tag{157}$$

$$e^{-\pi x^2} \supset e^{-\pi s^2} \tag{158}$$

$$1 \supset \delta(s) \tag{159}$$

$$\cos(\pi x) \supset I\!I(s) \tag{160}$$

$$\sin(\pi x) \supset i I_1(s) \tag{161}$$

$$I_1(x) \supset i \sin(\pi s) \tag{162}$$

Fourier Transforms exhibiting nonsymmetric properties

$$e^{i\pi x} \supset \delta(s - \frac{1}{2}) \tag{163}$$

$$\delta(x - \frac{1}{2}) \supset e^{-i\pi s} \tag{164}$$

Similarity theorem:

If f(x) has the Fourier Transform F(s) then f(ax) has the Fourier transform

$$\frac{1}{|a|} F(\frac{s}{a}) \tag{165}$$

A more symmetrical version of this theorem is(As each function expands or contracts it shrinks and grows vertically and the advantage is that the integral of the square remains constant in the power theorem):

If f(x) has the Fourier Transform F(s) then $|a|^{1/2}\, f(ax)$ has theFourier transform

$$\frac{1}{|a|^{1/2}} F(\frac{s}{a}) \tag{166}$$

Shift theorem:

If f(x) has the Fourier Transform F(s) then f(x-a) has theFourier transform

$$e^{-i2\pi as} F(s) \tag{167}$$

Power or Parseval's theorem:

$$\int_{-\infty}^{\infty} f(x)\, g^{*}(x)dx = \int_{-\infty}^{\infty} F(s)\, G^{*}(s)ds \tag{168}$$

Rayleigh's theorem:

$$\int_{-\infty}^{\infty} | f(x) |^2\, dx = \int_{-\infty}^{\infty} | F(s) |^2\, ds \tag{169}$$

Fourier Series

For a g(x) such that the following integrals exist:

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} g(x)dx \tag{170}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} g(x)\cos(2\pi nfx)dx \tag{171}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} g(x)\sin(2\pi nfx)dx \tag{172}$$

the series

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos(2\pi nfx) + b_n \sin(2\pi nfx)) \tag{173}$$

converges to the limit

$$\frac{1}{2}[g(x+0) + g(x-0)] \tag{174}$$

Therefore,

$$\begin{aligned} a_n - ib_n &= \frac{2}{T}\int_{-T/2}^{T/2} g(x)e^{-i2\pi nx}dx \\ &= \frac{2}{T}\int_{-\infty}^{\infty} g(x)\Pi(\frac{x}{T})e^{-i2\pi nx}dx \end{aligned} \tag{175}$$

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