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# SOIL MOISTURE SIMULATION BY IMPROVED NUMERICAL METHOD

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## PREFACE

A very large fraction of the water falling as rain on the land surfaces of the earth or applied irrigation water moves through unsaturated soil during the subsequent processes of infiltration, drainage, evaporation, and the absorption of soil-water by plant roots. The water movements in the unsaturated zone, together with the water holding capacity of this zone, are very important for the water demand of the vegetation, as well as for the recharge of the ground water storage. A fair description of the flow in the unsaturated zone is also crucial for predictions of the movement of pollutants into ground water aquifers.

This report entitled 'Soil Moisture Simulation by Improved Numerical Method' is a part of the research activities of 'Ground Water Assessment' division of the Institute. The purpose of this study is to develop a soil moisture simulation model using an improved numerical method. The study has been carried out by Mr. Chandra Prakash Kumar, Scientist 'C' under the guidance of Dr. G. C. Mishra, Scientist 'F'.

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## ABSTRACT

The theory for transient isothermal flow of water into nonswelling unsaturated soil is well understood and has been developed to a large extent in terms of solutions of the non-linear Richards equation. In the field, the description of infiltration is highly complicated since the initial and boundary conditions are usually not constant while the soil characteristics may vary with time and space. In view of this, most efforts in recent past, have been concentrated on seeking numerical solutions.

There exist quite a variety of finite difference solutions employing different forms of the non-linear Richards equation and different ways of discretization, the most common being explicit, implicit, and Crank-Nicolson approximation. The explicit approximation is derived by replacing the derivatives by their finite difference analog at the j time level. The implicit scheme replaces the derivatives by their finite difference analog at the (j+1) time level. The Crank-Nicolson approximation averages the derivatives at the j and (j+1) time levels to obtain an approximation at the (j+1/2) level implying that 50 % weightages are assigned to each of the derivatives at the j and (j+1) time levels.

The purpose of this study is to develop a numerical model to simulate the soil moisture profile in an initially unsaturated soil during infiltration. A model has been formulated for finite difference solution of the non-linear Richards equation applicable to transient, one-dimensional water flow through the unsaturated porous medium. The modification of explicit, implicit, and Crank-Nicolson schemes has been examined by varying the weightages assigned to the derivatives at the j and (j+1) time levels and comparing the simulated soil moisture profiles with the quasi-analytical solution of Philip.

## 1.0 INTRODUCTION

Most of the processes involving soil-water interactions in the field, and particularly the flow of water in the rooting zone of most crop plants, occur while the soil is in an unsaturated condition. Unsaturated flow processes are in general complicated and difficult to describe quantitatively, since they often entail changes in the state and content of soil water during flow. Such changes involve complex relations among the variable soil wetness, suction, and conductivity, whose inter-relations may be further complicated by hysteresis. The formulation and solution of unsaturated flow problems very often require the use of indirect methods of analysis, based on approximations or numerical techniques. For this reason, the development of rigorous theoretical and experimental methods for treating these problems was rather late in coming. In recent decades, however, unsaturated flow has become one of the most important and active topics of research and this research has resulted in significant theoretical and practical advances.

Richards (1931) presented the differential equation for soil water flow using an analogy to heat flow in porous media. Up to now this equation is used as the basic mathematical expression that underlies unsaturated flow phenomena. Soil water flow, however, is highly non-linear, as both the hydraulic conductivity and the soil water pressure head depend on the soil water content. Exact analytical solutions are only possible for simplified flow cases under a number of restrictive assumptions. Numerical solution of the flow equation on the other hand offers a powerful tool in approximating the real nature of the unsaturated zone for a wide variety of soil systems and external conditions.

The partial differential flow equation can be interpreted numerically by a finite difference, a finite element or a boundary element technique. Then a discretization scheme is applied for a system of nodal points that is superimposed on the soil depth-time region under consideration. Implementing the appropriate initial and boundary conditions then leads to a set of (linear) algebraic equations that can be solved by different methods. The operation by means of such a mathematical model is termed simulation, while the model is called simulation model.

The output of a simulation model can include such variables as pressure head, moisture content and flux as a function of soil depth and time. However, most frequently one calculates the terms of the water balance, i.e. infiltration, actual evaporation, actual transpiration, change in soil water storage and the net flux through the region boundary.

The main purpose of using dynamic simulation models is to assess the effects of water management measures such as irrigation, drainage, soil improvement and regional water supply plans, on the terms of the water balance of agricultural as well as nature conservation areas. Through the water balance terms one is generally able to evaluate effects of water management on e.g. crop yield and agricultural income. Transport of solutes is another aspect, which is directly related to the simulation of unsaturated water flow, i.e. the evaluation of pollution of the ground water reservoir, salinization, etc.

The yield of a crop well supplied with nutrients is directly related to its water use i.e. to its transpiration. The higher the water use, the higher the yield. Hence simulation of different irrigation regimes by a soil water balance model that has been combined with a crop growth model enables one to find the optimum regime. In such a case the crop and soil system should interact with each other, i.e. crop development with time should have a feedback with calculated actual water use and production rates.

The classical Richards-flow theory, upon which most simulation models are based, holds for stable flow conditions only. Yet instability of flow has been observed under a wide variety of circumstances such as abrupt and gradual increases of hydraulic conductivity with depth, compression of air ahead of the wetting front and water repellency of the solid phase. Another example of non-Richards type of flow is the preferential flow through non-capillary macropores. With classical flow theories one may then underestimate the velocity and depth of water/solute transport.

In the present study, a numerical model has been developed to simulate the soil moisture profile in an initially unsaturated soil during infiltration. The results are compared with the quasi-analytical solution of Philip (1957) for various sets of weightages assigned to the derivatives at the j and (j+1) time levels in the finite difference solution of the non-linear Richards equation.

## 2.0 REVIEW

### 2.1 General

Modelling infiltration in soils has been approached in the past in two main ways. In the first approach, hydrologists have recognized the difficulties associated with the solution of physically-based unsaturated flow equations and have opted for a large variety of empirical expressions with parameters to calibrate in optimization procedures. This approach, which has produced acceptable results for surface hydrologic computations, has not generated much understanding on the phenomenon of infiltration and distribution of water in unsaturated soils. In the second approach, it has been attempted to produce solutions to physically-based equations describing horizontal or vertical infiltration in soils. Several quasi-analytical solutions of the non-linear unsaturated flow equation have been reported in the literature. Other solutions use a numerical algorithm to implement in a computer.

### 2.2 Analogue Simulation Models

A hydrological simulation model is defined as "each system that can duplicate the response of a hydrological system". Simulation models which resemble the real world most closely are physical models (scale models) like for example sand tanks.

Analogue models are based on the similarity between the relations describing water dynamics and those describing physical phenomena such as electrical flow. Analogue models have the advantage of continuous simulation and they give a good approximation of the exact solution provided that the proper scale factors or transform functions are used. The main disadvantage is the time-consuming construction and operation. At this moment analogue simulation of water flow in the unsaturated zone is rarely applied. However, in combination with digital computers (hybrid models) most of the drawbacks can be overcome.

## 2.3 Mathematical Models

The dynamics of soil water is cast in the form of mathematical expressions that describe the hydrological relations within the system. The governing equations define a mathematical model. The entire model has usually the form of a set of partial differential equations, together with auxiliary conditions. The auxiliary conditions must describe the system's geometry, the system parameters, the boundary conditions and, in case of transient flow, also the initial conditions. Operations with such a mathematical model are called simulation.

If the governing equations and auxiliary conditions are simple, an exact analytical solution may be found. Otherwise, a numerical approximation is applicable. The numerical simulation models are by far the most applied ones.

### 2.3.1 Analytical approach

The relationships that govern the flow of water in unsaturated soil are quasilinear equations of the parabolic type. Since the coefficients in these equations are functions of the dependent variables, exact analytical solutions for specific boundary conditions are extremely difficult to obtain.

Analytical methods to solve the non-linear governing equations search for the exact solution in terms of analytical functions. Such an exact solution, if it exists, requires transformation, separation of variables, and usually a series of error functions.

The commonly used Boltzman transformation reduces the partial differential equations to ordinary differential equations. The Laplace transformation results in removing the time variable. The solution of an equation modified in this way yields a dependent variable as a function of the space variables. The non-linear mass conservation equation can be analytically solved only using various types of relaxation techniques such as linearization, quasilinearization and transformation to steady state.

The basic equation that describes one-dimensional vertical water movement in isotropic nonswelling soils with no consideration of sinks/sources can be derived by combining Darcy's law and the equation of continuity as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ K(\theta) \left( \frac{\partial h}{\partial z} - 1 \right) \right] \quad \text{...(2.1)}$$

where $\theta$ is soil moisture content, t is time, $K(\theta)$ is the hydraulic conductivity, h is the soil water pressure head and z is the gravitational head considered positive in downward direction. By introducing the soil water diffusivity $D(\theta) = K(\theta)/C(\theta)$, equation (2.1) can be written as :

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ D(\theta) \frac{\partial \theta}{\partial z} \right] - \frac{\partial K(\theta)}{\partial z} \quad \text{...(2.2)}$$

The following simplifications can be introduced to find analytical solutions : K is an analytical function of $\theta$ or h; hysteresis is neglected; the medium is homogeneous and isotropic; the flow is considered to be stationary or a succession of steady-state situations (quasistationary approach); the gravity force is neglected.

The first two assumptions linked with the third one have resulted in a great number of analytical solutions. The gravity force is often neglected in describing the infiltration process in originally dry soil, resulting in analytical solutions as derived by e.g. Philip (1957, 1958).

### 2.3.2 Numerical approach

With the advance of digital computers, emphasis has shifted drastically from the classical approach of analytical solutions to the rapidly developing field of numerical analysis. At present, numerical approximations are possible for complex, compressible, nonhomogeneous and anisotropic flow regions having various boundary configurations.

Numerical methods are based on subdividing the flow region into finite segments bounded and represented by a series of nodal points at which a solution is obtained. This solution depends on the solutions of the surrounding segments and on an appropriate set of auxiliary conditions.

In recent years a number of numerical methods have been introduced. The methods most appropriate to the problem of soil water dynamics are finite difference method, finite element method and boundary element method. The finite difference method has been discussed below.

## Finite difference methods

Finite difference methods (Remson et al., 1971), either explicit or implicit, belong to the most frequently used techniques in modelling unsaturated flow conditions. The most simple type of finite differencing, the explicit one, orders the differencing operators in such a manner that the resulting finite difference equation contains only one unknown, and consequently, may be solved simply and directly. The explicit method is computationally simple but it has one serious drawback. In order to attain reasonable accuracy, the length of the interval in space must be kept small. To get a stable solution, the time step has to be small compared with the space interval. Thus it is necessary to have a large number of time steps when using the simple explicit method.

Implicit solution methods generally use much larger time steps than explicit ones, but their stability depends upon the degree of nonlinearity of the differential equation. There are a great number of methods to solve an implicit set of algebraic equations, such as linearization, predictor-corrector or iteration methods.

In dealing with unsaturated flow problems that involve more than one space dimension and a grid with many nodal points, it is often necessary to use a mixed scheme that relies on simultaneous displacements along one space dimension and on successive displacements along the remaining space dimensions. This leads to the method of successive over relaxation (SOR). In the case of isotropic conditions, faster convergence may be sometimes achieved by using the iterative alternating direction implicit procedure (ADIPIT).

The advantage of the finite difference method is its simplicity and efficiency in treating the time derivatives. On the other hand, the method is rather incapable to deal with complex geometries of flow regions. A slow convergence, a restriction to bilinear grids and difficulties in treating moving boundary conditions are other serious drawbacks of the method.

## 2.4 Initial and Boundary Conditions

Initial conditions must be defined when transient soil water flow is modelled. Usually values of matric head or soil water content at each nodal point within the soil profile are required. However, when these data are not available, water contents at field capacity or those in equilibrium with the ground water table might be considered as the initial ones.

### 2.4.1 Upper boundary conditions

While the potential evapotranspiration rate from a soil depends only on crop and atmospheric conditions, the actual flux through the soil surface and the plants is limited by the ability of the soil matrix to transport water. Similarly, if the potential rate of infiltration exceeds the infiltration capacity of the soil, part of the water runs off, since the actual flux through the top layer is limited by moisture conditions in the soil. Consequently, the exact boundary conditions at the soil surface can not be estimated a priori and solutions must be found by maximizing the absolute flux. The minimum allowed pressure head at the soil surface, $h^{lim}$ (time dependent) can be determined from equilibrium conditions between soil water and atmospheric vapour.

The possible effect of ponding has been neglected so far. In case of ponding, usually the height of the ponded water as a function of time is given. However, when the soil surface is at saturation then the problem is to define the depth in the soil profile where the transition from saturation to partial saturation occurs.

In most of the dynamic transient models, the surface nodal point is treated during the first iteration as a prescribed flux boundary and matric head h is computed. If $h^{lim} \leq h \leq 0$, the upper boundary condition remains a flux boundary during the whole iteration. If not, the surface nodal point is treated as a prescribed pressure head in the following iteration. Then in case of infiltration, h = 0 and in case of evaporation $h = h^{lim}$. The actual flux is then calculated explicitly and is subject to the condition that actual upward flux through the soil-air interface is less than or equal to potential evapotranspiration (time dependent).

### 2.4.2 Lower boundary conditions

At the lower boundary one can define three different types of conditions : (a) Dirichlet condition, the pressure head is specified; (b) Neumann condition, the flux is specified; and (c) Cauchy condition, the flux is a function of a dependent variable.

The phreatic surface (place, where matric head is atmospheric) is usually taken as lower boundary of the unsaturated zone in the case where recorded water table fluctuations are known a priori. Then the flux through the bottom of the system can be calculated. In regions with a very deep ground water table, a Neumann type of boundary condition is used.

### Dirichlet condition

Easy recording of changes in phreatic surface in case of present ground water table is the main advantage of specifying a matric head zero as the bottom boundary. A drawback is that with shallow ground water tables (less than 2 m below soil surface) the simulated effects of changes in phreatic surface are extremely sensitive to variations in the soil hydraulic conductivity.

The nodal points in a soil profile usually have fixed positions and probably none of them will coincide with the water table level. The nodal point, where the matric head is prescribed, is often the one immediately beneath the phreatic level. When large fluxes across the lower boundary occur, an error is introduced by this approximation.

### Neumann condition

A flux as lower boundary condition is usually applied in cases where one can identify a no-flow boundary (e.g. an impermeable layer) or a free drainage case. In the latter case the flux is always directed downward and the gradient $\partial h/\partial z = 1$, so the Darcian flux is equal to the hydraulic conductivity at the lower boundary.

### Cauchy condition

This type of boundary condition is used when unsaturated flow models are combined with models for regional ground water flow or when the effects of surface water management are to be simulated. Writing the lower boundary flux as function of the phreatic surface, which is in this case the dependent variable, one can incorporate relationships between the flux to/from the drainage system and the height of the phreatic surface. This flux-head relation can be obtained from drainage formulae or from regional ground water flow models.

With the lower boundary conditions the connection with the saturated zone can be established. In this way effects of activities influencing the regional ground water system upon, for instance, crop evapotranspiration can be simulated. The coupling between the two systems is possible by considering the phreatic surface as an internal moving boundary with one-way or two-way relationships. When the Cauchy condition is linked with a one-dimensional vertical flow model, one can consider such a solution as quasi-two-dimensional, since both vertical and horizontal flow are calculated.

## 2.5 Required Input Data

Simulation of water dynamics in the unsaturated zones requires input data concerning the model parameters, the geometry of the system, the boundary conditions and, when simulating transient flow, initial conditions. With geometry parameters the dimensions of the problem domain are defined. With the physical parameters the physical properties of the system under consideration are described. With respect to the unsaturated zone it concerns the soil water characteristic, $\theta(h)$, and the hydraulic conductivity, $K(\theta)$. If root water uptake is also modelled, the parameters defining the relation between root water uptake and soil water status should be given, together with crop specifications. In case a functional flux-head relationship is used as lower boundary condition, the parameters describing the interaction between surface water and ground water and, if necessary, the vertical resistance of poorly permeable layers have to be supplied.

## 2.6 Hysteresis

Most simulation models ignore the effect of hysteresis. However, it has been recognized for a long time that hysteresis in the soil water retention curve influences the soil water movement, especially when frequent changes from wetting to drying occur. The hysteresis phenomenon does not affect the $K(\theta)$ relation very much and is usually neglected. It may be noted that if $K(\theta)$ functions are not taken as subject to hysteresis, they might be considered being dependent on temperature.

The main reasons for hysteresis in the water retention curve are the complexity of the pore-space geometry, the presence of entrapped air, shrinking and swelling and thermal gradients. The first mathematical models of hysteresis were based on the so-called independent domain concept. The basic assumptions of this concept are (1) a difference in the water volume of each pore does not depend on matric head and (2) the pore space is built up of pores or domains with each pore size defined by two matric heads. Later on, models were developed by introducing a domain dependence factor.

However, successful attempts to build the hysteresis problem into dynamic simulation transient flow models are still scarce. Some of the practical problems, e.g. low pressure/very frequent irrigation or significant shrinkage/swelling clearly define a range of flow situations, where hysteresis can not be omitted. For the case when water retention is noticeably affected by heavy swelling/shrinking and water adsorption-desorption, the boundary hysteresis curves do not join even for the highest values of matric heads. Under such conditions all of the present hysteresis models still need adaptations.

## 2.7 Preferential Flow

Most simulation models for the unsaturated zone consider the soil to be isotropic and homogeneous. The fact that most soils are neither was recognized already in the 19th century. In field soils, transport of water is often heterogeneous with part of the infiltrating water travelling faster than the average wetting front. This has important consequences for simulating the field water balance and therefore on the calculation of crop water use, crop yield, solute transport and pollution of ground water and subsoil. In some soils, preferential flow occurs through large pores in an unsaturated soil matrix, a process known as bypass flow or shortcircuiting. In other soils, different flow rates vary more gradually, while matrix and preferential pathways can not be distinguished easily.

Preferential flow of water through unsaturated soil can be caused by different mechanisms, one of them being the occurrence of noncapillary sized macropores. This type of macroporosity can be caused by shrinking and cracking of the soil, by plant roots, by soil fauna or by tillage operations. The occurrence of wetting front instability, as caused by air entrapment ahead of the wetting front or by water repellency of the soil can also be viewed as an expression of preferential flow. Whatever the cause of preferential flow, the result is that the basic partial differential equation describing flow within the soil matrix domain, needs adaptation.

The partitioning of soil water over the soil matrix and macropores, and the fate of water flowing downward through the macropores is handled differently by the various models. The common principle, however, is essentially the two-domain concept. An important aspect of preferential flow is the interaction between water in the soil matrix and water inside the macropores. In some models, the total preferential flow is accumulated at the bottom of the macropores and is then added to the unsaturated zone at that depth.

## 2.8 Spatial Variability and Scaling

Most models for the unsaturated zone are one-dimensional. However, the problems which have to be modelled, are in general of local or regional nature. In that case, we face the problem of spatial variability. This phenomenon recently has attracted much attention in literature. The basic assumption is that the porous medium is regarded as a macroscopic continuum with properties that are continuous functions of the space coordinates. A set of measured values is interpreted as a realization of a spatial stochastic function. The estimation of these functions may be very complicated. Also the application of geostatistics with regionalization of point simulations is of value. A proper application of the geostatistical approach may reveal field characteristics that are not apparent from conventional statistical analysis.

A phenomenon connected with regional application of one-dimensional simulation models is scaling. In principle, scaling is a technique of expressing the statistical variability in, for instance, the hydraulic conductivity in handsome relationships. By this simplification, the pattern of spatial variability is described by a set of scale factors, defined as the ratio between the characteristic phenomenon at the particular location and the corresponding phenomenon of a reference soil.

## 3.0 PROBLEM DEFINITION

The one-dimensional problems of interest in subsurface hydrology are mainly non-linear and the auxiliary conditions of most natural systems are extremely complicated. For such situations, exact analytic solutions are not available and recourse must be made to the use of numerical methods of approximation for the solution of differential equations. Thus numerical methods are powerful tools for the solution of realistic mathematical models of complicated natural subsurface hydrologic systems. The methods most appropriate to the problem of soil water dynamics are finite difference method, finite element method and boundary element method. In the present study, finite difference method has been used.

The basic idea of finite difference methods is to replace derivatives at each of a number of mesh points by ratios of the changes of appropriate variables over a small but finite interval. This reduces a continuous boundary-value problem to a set of algebraic equations.

The 'explicit' approximation is derived by replacing the space derivative by its finite difference analog at the j time level. The time derivative is then replaced by an approximation between the j and (j+1) time levels. Applied at each mesh point, the approximation contains one unknown value of the dependent variable at the (j+1) time level and can be solved explicitly in terms of the three known values at the j time level. The finite difference computation must be convergent and stable to give a result that is in some sense close to the solution of the original problem. Explicit computations require many small time jumps and therefore implicit approximations are preferable.

The 'implicit' formulation is obtained by replacing the space derivative with its finite difference analog at the (j+1) time level. The time derivative is then replaced by a backward-difference approximation (relative to the j+1 time level). Thus, the approximation at each node involves three unknown values at the (j+1) level and one known value at the j time level. When applied at all nodes, the result is a system of simultaneous linear algebraic equations with unknowns at the (j+1) time level. With boundary conditions specified, the number of equations equals the number of unknowns, and the system can be solved to 'march' the solution forward one time step. The coefficients of these equations form a tridiagonal matrix and the system of equations is therefore easy to solve.

The Crank-Nicolson approximation averages the space derivative at the j and (j+1) levels to obtain an approximation at the (j+1/2) level and the usual approximation for the time derivative over the interval from j to (j+1). The Crank-Nicolson approximation gives a system of equations for the boundary-value problem that retains the computationally advantageous tridiagonal form and has a small truncation error.

The finite difference methods make use of approximations. However, the resulting inaccuracies can be made negligibly small through proper use of the methods. Furthermore these errors in approximation are generally outweighed by the inaccuracies due to the uncertainties in the specification of subsurface hydrologic parameters.

The objective of the present study is to develop a numerical model (finite difference scheme) for solving the non-linear partial differential equation (Richards equation) describing one-dimensional water flow through the unsaturated porous medium and to examine the modification of explicit, implicit and Crank-Nicolson methods by varying the weightages assigned to the derivatives at the j and (j+1) time levels. The simulated soil moisture profiles at various times in a sandy soil have been compared with the soil moisture profiles obtained through quasi-analytical solution of Philip. The Philip's quasi-analytical solution was obtained by solving the Richards equation subject to condition of a constant pressure at the soil surface (Haverkamp et al., 1977).

## 4.0 METHODOLOGY

### 4.1 General

Due to the non-linearity of the equations describing the flow under unsaturated conditions, the solution in general requires numerical methods. Analytical solutions are known for special cases only. As the analytical methods are difficult to apply in most cases, numerical approximation methods are often utilized for solving differential equations. Among the available numerical methods, those employing finite differences are most frequently used.

Finite difference methods are approximate in the sense that derivatives at a point are approximated by difference quotients over a small interval, but the solutions are not approximate in the sense of being crude estimates. Finite difference methods generally give solutions that are either as accurate as the data available or as accurate as necessary for the technical purposes for which the solutions are required.

### 4.2 General Equation of Unsaturated Flow

A proper physical description of water flow in the soil requires that three parameters be specified : flux, hydraulic gradient, and conductivity. Knowledge of any two of these allows the calculation of the third, according to Darcy's law. This law states that the flux equals the product of conductivity by the hydraulic gradient. Darcy's law has been found to apply for unsaturated as well as for saturated soils, but the pressure gradient at unsaturation becomes a suction gradient, and the hydraulic conductivity is no longer constant, but a function of water content or suction. Since the conductivity depends on the number, sizes, and shapes of the conducting pores, its value is greatest when the soil is saturated, and decreases steeply when the soil water suction increases and the soil loses moisture. Darcy's law suffices to describe water flow under steady state conditions, but must be combined with the continuity equation to describe unsteady (transient state) flow. According to Darcy's law, for one-dimensional vertical flow, the volumetric flux q ($cm^3/cm^2/h$) can be written as

$$q = - K \frac{\partial}{\partial z} (h - z) \qquad (cm/h)$$

or

$$q = - K \left(\frac{\partial h}{\partial z} - 1\right) \qquad (cm/h) \qquad \ldots(4.1)$$

where K is the hydraulic conductivity (cm/h), h is the soil water pressure head (relative to the atmosphere) expressed in cm of water and z is the gravitational head (cm) considered positive in downward direction.

In order to get a complete mathematical description for unsaturated flow, we apply the continuity principle (Law of Conservation of Matter)

$$\frac{\partial \theta}{\partial t} = - \frac{\partial q}{\partial z} \qquad (/h) \qquad \ldots(4.2)$$

where $\theta$ is soil moisture content expressed in $cm^3/cm^3$ and t is time in hours.

Substitution of equation (4.1) into equation (4.2) yields the partial differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ K \left(\frac{\partial h}{\partial z} - 1\right) \right] \qquad \ldots(4.3)$$

Equation (4.3) is a second order, parabolic type of partial differential equation (known as Richards equation) which is non-linear because of the dependency of K and h on $\theta$ (linearity means that the coefficients in a differential equation are only functions of the independent variables z and t). To avoid the problem of the two dependent variables $\theta$ and h, the derivative of $\theta$ with respect to h can be introduced, which is known as the specific water capacity C

$$C = \frac{d\theta}{dh} \qquad (/\mathrm{cm}) \qquad \ldots(4.4)$$

In equation (4.4) a normal instead of a partial derivative notation is used, because h is considered here as a single value function of $\theta$ (no hysteresis). Writing

$$\frac{\partial \theta}{\partial t} = \frac{d\theta}{dh} \cdot \frac{\partial h}{\partial t} \qquad \ldots(4.5)$$

and substituting equation (4.4) into equation (4.3) yields

$$C(h) \frac{\partial h}{\partial t} = \frac{\partial}{\partial z} \left[ K(h) \left(\frac{\partial h}{\partial z} - 1\right) \right] \qquad \ldots(4.6)$$

In equation (4.6) the coefficients C and K are functions of the dependent variable h, but not functions of the derivatives $\partial h/\partial t$ and $\partial h/\partial z$. Written in this form, equation (4.6) provides the basis for predicting soil water movement in layered soils of which each layer may have different physical properties.

## 4.3 Initial and Boundary Conditions

To obtain a solution for the one-dimensional vertical flow equation, equation (4.6) must be supplemented by appropriate initial and boundary conditions.

As initial condition (at $t = 0$) the pressure head is specified as a function of the depth z

$$h\ (z,\ t = 0) = h_0 \qquad \text{...(4.7)}$$

As hysteresis is not considered in this study, this condition is equivalent to

$$\theta\ (z,\ t = 0) = \theta_0 \qquad \text{...(4.8)}$$

One can then easily obtain the value of h (and vice versa) from the expression : $h = f(\theta)$.

To describe the boundary conditions one can distinguish between three types :

(a) Dirichlet condition : specification of the dependent variable, the pressure head

$$\left.\begin{array}{l} h\ (z = 0,\ t) = h_u \\ h\ (z = L,\ t) = h_l \end{array}\right\} \qquad \text{...(4.9)}$$

These conditions are equivalent to

$$\left.\begin{array}{l} \theta\ (z = 0,\ t) = \theta_u \\ \theta\ (z = L,\ t) = \theta_l \end{array}\right\} \qquad \text{...(4.10)}$$

(b) Neumann condition : specification of the derivative of the pressure head. For the soil water problem this condition means a specification of the flow through the boundaries

$$q(t) = - K(h) \left(\frac{\partial h}{\partial z} - 1\right) \qquad \ldots(4.11)$$

(c) 'Mixed' condition, a combination of the first two types. In particular this can specify h at the lower boundary and q at the upper boundary.

For the present study, initial condition has been defined by equation (4.8) as

$$\theta \ (z,\ t = 0) = 0.10 \qquad \ldots(4.12)$$

and the upper boundary condition by equation (4.10) as

$$\theta \ (z = 0,\ t) = 0.267 \qquad \ldots(4.13)$$

## 4.4 Soil Moisture Characteristics

For the present study, functional relations, as reported by Haverkamp et al. (1977), for characterizing the hydraulic properties of a soil, were used. They compared six models, employing different ways of discretization of the non-linear infiltration equation in terms of execution time, accuracy, and programming considerations. The models were tested by comparing water content profiles calculated at given times by each of the model with results obtained from an infiltration experiment carried out in the laboratory. All models yielded excellent agreement with water content profiles measured at various times.

The infiltration experiments were done in the laboratory using a plexiglass column, 93.5 cm long and 6 cm inside diameter uniformly packed with sand to a bulk density of 1.66 $gm/cm^3$. The column was equipped with tensiometers at depths of 7, 22, 37, 52, 67 and 82 cm below the soil surface. Each tensiometer had its own pressure transducer. The changes of water content at different depths were obtained by gamma ray attenuation using a source of Americium-241. A constant water pressure ($\theta$ = 0.10) was maintained at the lower end of the column, a constant flux (13.69 cm/h) was imposed at the soil surface (z = 0) and initial condition as $\theta$ = 0.10 throughout the depth. The hydraulic conductivity and water content relationship of the soil was obtained by analysis of the water content and water pressure profiles during transient flow. The soil water pressure and water content relationship was obtained at each tensiometer depth by correlating tensiometer readings and water content measurements during the experiments. The following analytical expressions, obtained by a least square fit through all data points were chosen for characterizing the soil :

$$K = K_s \frac{A}{A + |h|^{\beta_1}} ; \qquad ...(4.14)$$

where,

$K_s$ = 34 cm/h,
$A$ = $1.175 \times 10^6$,
$\beta_1$ = 4.74.

and $$\theta = \frac{\alpha(\theta_s - \theta_r)}{\alpha + |h|^{\beta_2}} + \theta_r ; \qquad ...(4.15)$$

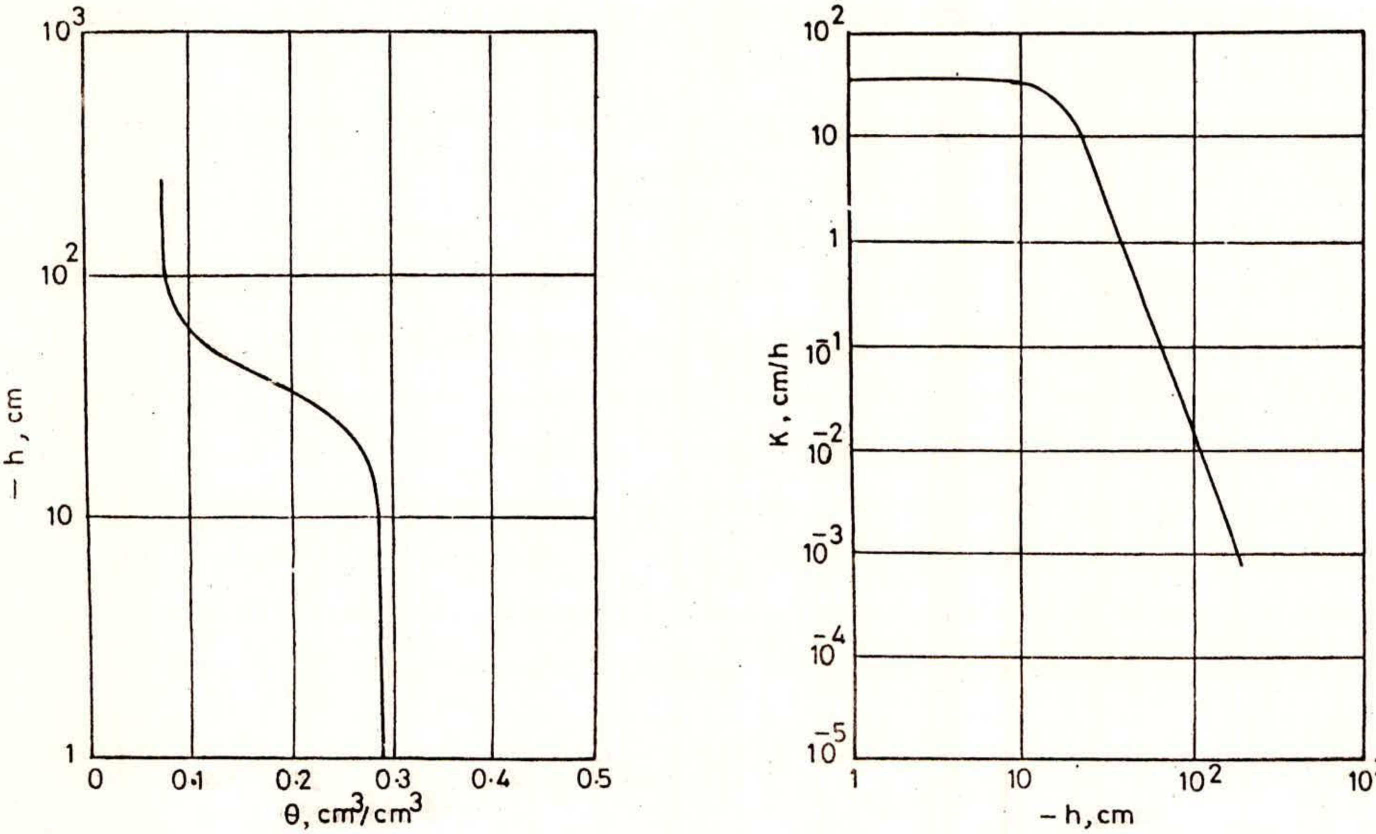


FIG.1. RELATIONSHIPS BETWEEN THE SOIL WATER PRESSURE h , THE WATER CONTENT θ AND THE HYDRAULIC CONDUCTIVITY K FOR THE SOIL USED IN THE STUDY

where,

$$\theta_s = 0.287,$$
$$\theta_r = 0.075,$$
$$\alpha = 1.611 \times 10^6,$$
$$\beta_2 = 3.96.$$

Subscript s refers to saturation, i.e. the value of $\theta$ for which $h = 0$, and the subscript r to residual water content.

Figure 1 present the relationships between the soil water pressure h, the water content $\theta$ and the hydraulic conductivity K for the above soil used in this study.

## 4.5 Finite Difference Approximation

Let the entire flow domain be divided into a grid of equal intervals $\Delta z$ and the time domain be similarly divided into intervals $\Delta t$. The partial differential equation (4.6) can be approximated by a finite difference equation replacing $\partial t$ and $\partial z$ by $\Delta t$ and $\Delta z$ respectively in the following way :

$$C_i^{j+a} \frac{h_i^{j+1} - h_i^j}{\Delta t} = \frac{1}{\Delta z} \left[ K_{i+1/2}^{j+a} \left( \frac{h_{i+1}^{j+a} - h_i^{j+a}}{\Delta z} - 1 \right) - K_{i-1/2}^{j+a} \left( \frac{h_i^{j+a} - h_{i-1}^{j+a}}{\Delta z} - 1 \right) \right] \quad \ldots(4.16)$$

where i and j are the indexes of space and time respectively. 'a' is a weighting factor ($0 \leq a \leq 1$) introduced in such a manner that by putting a = 0, it is transformed into explicit scheme, a = 0.5 into Crank-Nicolson scheme and $\varepsilon$ = 1 into implicit scheme. Therefore,

$$h_i^{j+a} = (1 - a)\, h_i^{j} + a\, h_i^{j+1} \qquad \ldots(4.17\ a)$$

$$h_{i+1}^{j+a} = (1 - a)\, h_{i+1}^{j} + a\, h_{i+1}^{j+1} \qquad \ldots(4.17\ b)$$

$$h_{i-1}^{j+a} = (1 - a)\, h_{i-1}^{j} + a\, h_{i-1}^{j+1} \qquad \ldots(4.17\ c)$$

The values of $C_i^{j+a}$, $K_{i+1/2}^{j+a}$ and $K_{i-1/2}^{j+a}$ can be approximated by

$$C_i^{j+a} = F_1 = (1 - a)\, C_i^{j} + a\, C_i^{j+1} \qquad \ldots(4.18\ a)$$

$$K_{i+1/2}^{j+a} = F_2 = (1 - a)\, K_{i+1/2}^{j} + a\, K_{i+1/2}^{j+1}$$

$$= (1 - a) \sqrt{(K_i^{j}\, K_{i+1}^{j})} + a \sqrt{(K_i^{j+1}\, K_{i+1}^{j+1})} \qquad \ldots(4.18\ b)$$

$$K_{i-1/2}^{j+a} = F_3 = (1 - a)\, K_{i-1/2}^{j} + a\, K_{i-1/2}^{j+1}$$

$$= (1 - a) \sqrt{(K_{i-1}^{j}\, K_i^{j})} + a \sqrt{(K_{i-1}^{j+1}\, K_i^{j+1})} \qquad \ldots(4.18\ c)$$

Different methods of weighting interblock hydraulic conductivity values for modelling one-dimensional water transfer in homogeneous unsaturated soil were tested by Haverkamp and

Vauclin (1979) for their influence upon the accuracy of the finite difference solution. The only method that well simulated their experimental observations was the geometric mean. This approach has therefore been adopted in the equations (4.18 b) and (4.18 c).

Substitution of equations (4.17) and (4.18) in equation (4.16) yields the following linear algebraic equation valid for each nodal point :

$$\left[ - a F_3 \frac{\Delta t}{(\Delta z)^2} \right] h_{i-1}^{j+1} + \left[ F_1 + a F_2 \frac{\Delta t}{(\Delta z)^2} + a F_3 \frac{\Delta t}{(\Delta z)^2} \right] h_i^{j+1}$$

$$- \left[ a F_2 \frac{\Delta t}{(\Delta z)^2} \right] h_{i+1}^{j+1} = \left[ (1 - a) F_3 \frac{\Delta t}{(\Delta z)^2} \right] h_{i-1}^{j}$$

$$+ \left[ F_1 - (1 - a) F_2 \frac{\Delta t}{(\Delta z)^2} - (1 - a) F_3 \frac{\Delta t}{(\Delta z)^2} \right] h_i^{j}$$

$$+ \left[ (1 - a) F_2 \frac{\Delta t}{(\Delta z)^2} \right] h_{i+1}^{j} + (F_3 - F_2) \frac{\Delta t}{\Delta z} \qquad \text{...(4.19)}$$

When equation (4.19) is applied at all nodes, the result is a system of simultaneous linear algebraic equations with a tridiagonal coefficient matrix with zero elements outside the diagonals and unknown values of h. In solving this system of equations, a so-called direct method was used by applying a tridiagonal algorithm of the kind discussed by Remson et al.. (1971).

## 4.6 Soil Moisture Simulation

Due to the fact that the hydraulic conductivity, K(h) and specific water capacity, C(h) at the end of each time step are unknown, an iterative process was used. In the iteration method, $h_i^j$ is replaced by $h_i^j(n)$, where n is an iteration index. For the first iteration, $h_i^j(0)$ is set equal to $h_i^j$. The resulting linear equations are then solved for $h_i^{j+1}$, and $h_i^j(1)$ is obtained from this solution. The parameters C(h) and K(h) are adjusted corresponding to this estimate of $h_i^j$ and the equations are solved again to find $h_i^j(2)$, and the procedure is continued. The iterative procedure is generally terminated when two successive values of $h_i^j$ are 'close' to each other e.g.

$$\sum_{i=1}^{nnode} [\, h_i^j(n) - h_i^j(n-1) \,]^2 \leq 0.0001 \qquad \ldots(4.20)$$

The iteration method is time-consuming, but gives better estimates.

A specific solution of Richards equation was obtained by Philip (1957) in the case of infiltration in an homogeneous semi-infinite column satisfying the boundary conditions :

$$\left.\begin{array}{lll} t < 0 & z \geq 0 & \theta = \theta_0 \\ t \geq 0 & z = 0 & \theta = \theta_u \end{array}\right\} \qquad \ldots(4.21\ a)$$

In a later paper (Philip, 1958), the Richards equation was solved for the conditions :

$$\left.\begin{array}{lll} t < 0 & z \geq 0 & h = h_0 \\ t \geq 0 & z = 0 & h = h_u \end{array}\right\} \qquad \ldots(4.21\ b)$$

where $h_u$ could take positive values corresponding to an infiltration experiment with submersion. Philip's method led to a solution in the form of a power series in $t^{1/2}$. Since the series converges only for finite t, the solution becomes unreliable as t tends to infinity; the t-range of convergence is depending upon the characteristics of soil and the initial and boundary conditions.

In the present study, soil moisture profiles were simulated at various times for values of weighting factor 'a' ranging from 0 to 1 (with an increment of 0.05) and compared with the quasi-analytical solution of Philip by computing an error term over the zone of interest, as follows :

$$\text{Error Term} = \sum_{i=m}^{i=m+n} [\, \theta_i^j(\text{simulated}) - \theta_i^j(\text{Philip})\, ]^2 \qquad \ldots(4.22)$$

The values of m and n depend upon the advancement of water front at various times.

The computer code, for discretization scheme used in the model and simulation of soil moisture profiles as per the procedure described above, has been written in FORTRAN and presented in Appendix - I.

## 5.0 RESULTS

The numerical model described in section 4.5 was tested for different values of weighting factor 'a' by comparing water content profiles calculated at given times with results obtained from quasi-analytical solution of Philip. Using the functional relations given in equations (4.14) and (4.15) for characterizing the hydraulic properties of the soil, the water content profiles were determined subject to the following conditions :

$$\left.\begin{array}{lll} t < 0 & z \geq 0 & \theta_0 = 0.10 \text{ cm}^3/\text{cm}^3 \\ t \geq 0 & z = 0 & \theta_u = 0.267 \text{ cm}^3/\text{cm}^3 \end{array}\right\} \quad \ldots(5.1)$$

The numerical computations were made with a depth interval $\Delta z$ = 1 cm, the total simulation period being 0.8 hour. It was found by trial and error that the numerical scheme is stable for time step $\Delta t$ = 0.4 second in case of a = 0 (i.e. explicit scheme) and $\Delta t$ = 5 seconds in case of a = 1 (i.e. implicit scheme). In order to examine the effect of weighting factor 'a' on the simulated water content profiles, the time step was kept constant as 0.4 second for all simulations.

Haverkamp et al. (1977) has reported the infiltration profiles at various times for infiltration in the sand (under consideration) obtained by quasi-analytical solution of Philip. Numerical data of Philip's solution are given in table 1. For 'limited times' Philip's method gives at each time, t the depth, z(t) which reaches a given water content, $\theta_i$ according to;

$$z(t,\theta_i) = \phi(\theta_i)\, t^{1/2} + \chi(\theta_i)\, t + \psi(\theta_i)\, t^{3/2}$$

$$+ \omega(\theta_i)\, t^{2} + \ldots\ldots\ldots + f_n(\theta_i)\, t^{n/2} \qquad \ldots(5.2)$$

Table 1 : Water Content Profiles determined with the Solution of Philip

| Water Content ($\theta$) | Depth (z) | | |
|---|---|---|---|
| | t = 0.1 hour | t = 0.2 hour | t = 0.8 hour |
| 0.2523 | 9.4 | 17.7 | 65.2 |
| 0.2356 | 12.0 | 20.7 | 69.2 |
| 0.2189 | 13.2 | 22.1 | 71.1 |
| 0.2021 | 14.1 | 23.1 | 72.3 |
| 0.1854 | 14.8 | 23.8 | 73.2 |
| 0.1686 | 15.3 | 24.5 | 74.0 |
| 0.1519 | 15.9 | 25.2 | 74.8 |
| 0.1351 | 16.5 | 25.9 | 75.7 |
| 0.1184 | 17.3 | 26.8 | 76.8 |
| 0.1016 | 19.5 | 29.5 | 78.6 |

The numerical model, on the other hand, calculates $\theta$ for a given value of z. As a result, interpolations are necessary, at a given stage of calculations to compare the results. The prediction of the water content profiles using Philip's method is only valid within the domain of convergence of the series (equation 5.2). To calculate the time for which the series would converge, Philip (1969, pp. 250) introduced a characteristic time of infiltration, $t_{grav}$, as :

$$t_{grav} = \left[ \frac{S}{K_u - K_0} \right]^2 \qquad \ldots(5.3)$$

where, $K_u$ is the hydraulic conductivity corresponding with $\theta_u$ and S is the sorptivity, defined as

$$S = \int_{\theta_0}^{\theta_u} \phi \, d\theta \qquad \ldots(5.4)$$

For the sand material, S was found to be 5.441 $cm/h^{1/2}$, and the characteristic time was $t_{grav}$ = 0.16 hour. Consequently for $t \leq 0.2$ hour, the water content profile could be calculated with equation (5.2). In our calculations the series was limited to four terms. To use more terms of the series would, according to Philip, 'extend the range of accurate results only by small amounts quite disproportionate to the extra labour involved'. For $t \geq 0.3$ hour, the profiles were calculated by an approximation of the 'infinite' profile, as proposed by Philip (1957, pp. 444). The power series solution (equation 5.2) and the asymptotic solution of the profile at infinity are expected to overlap.

Table 2 present the 'error term' for simulated water content profiles for different values of 'weighting factor' as compared with the quasi-analytical solution of Philip at t = 0.1 hour, 0.2 hour and 0.8 hour. The variation of error term with respect to weighting factor has also been presented in figures 2, 3 and 4 for t = 0.1 hour, 0.2 hour and 0.8 hour respectively. It can be observed that the error term increases with time irrespective to the value of 'a'. Secondly, the error term continuously decreases as the weighting factor increases from 0 to 1 for all the three times of simulations. It implies that the implicit scheme (a = 1) gives better agreement with infiltration profiles calculated with Philip's method instead of any other value of 'a' including a = 0 (explicit scheme) and a = 0.5 (Crank-Nicolson scheme).

Table 2 : Error Term for Simulated Water Content Profiles

| S.No. | Weighting Factor (a) | Error Term | | |
|---|---|---|---|---|
| | | t = 0.1 hour | t = 0.2 hour | t = 0.8 hour |
| 1 | 0.00 | 0.000256764 | 0.000390260 | 0.003107394 |
| 2 | 0.05 | 0.000255853 | 0.000389305 | 0.003101068 |
| 3 | 0.10 | 0.000255068 | 0.000388679 | 0.003101077 |
| 4 | 0.15 | 0.000254360 | 0.000388216 | 0.003103563 |
| 5 | 0.20 | 0.000253568 | 0.000387457 | 0.003100830 |
| 6 | 0.25 | 0.000252738 | 0.000386601 | 0.003095082 |
| 7 | 0.30 | 0.000252007 | 0.000385968 | 0.003093199 |
| 8 | 0.35 | 0.000251332 | 0.000385481 | 0.003093848 |
| 9 | 0.40 | 0.000250595 | 0.000384782 | 0.003091039 |
| 10 | 0.45 | 0.000249918 | 0.000384144 | 0.003087688 |
| 11 | 0.50 | 0.000249275 | 0.000383572 | 0.003086450 |
| 12 | 0.55 | 0.000248619 | 0.000382972 | 0.003083668 |
| 13 | 0.60 | 0.000247968 | 000382360 | 0.003081211 |
| 14 | 0.65 | 0.000247388 | 000381849 | 0.003079181 |
| 15 | 0.70 | 0.000246836 | 0.000381409 | 0.003078590 |
| 16 | 0.75 | 0.000246251 | 0.000380848 | 0.003075449 |
| 17 | 0.80 | 0.000245728 | 0.000380369 | 0.003073623 |
| 18 | 0.85 | 0.000245203 | 0.000379933 | 0.003072237 |
| 19 | 0.90 | 0.000244702 | 0.000379518 | 0.003071164 |
| 20 | 0.95 | 0.000244258 | 0.000379155 | 0.003070244 |
| 21 | 1.00 | 0.000243760 | 0.000378676 | 0.003068143 |

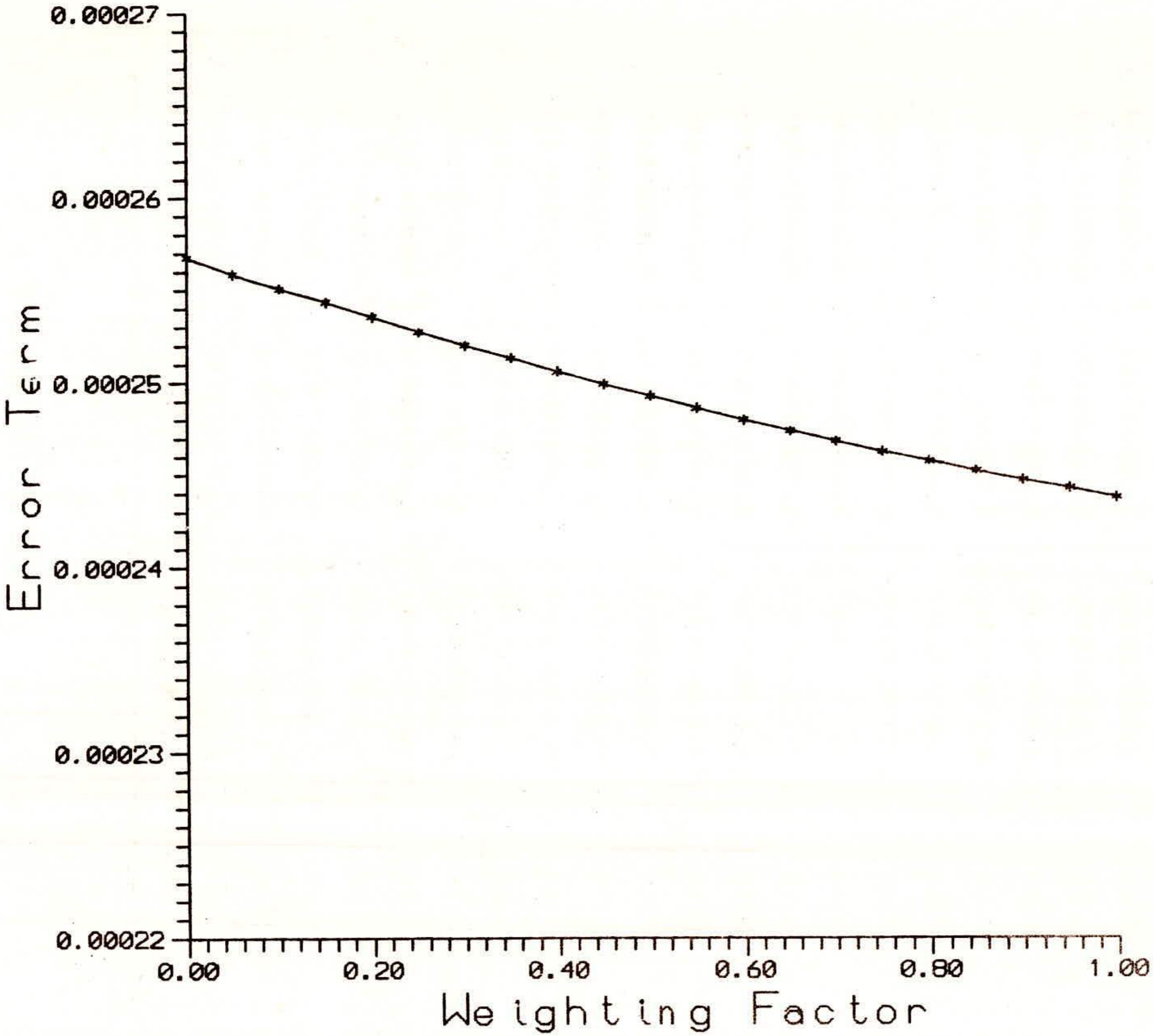


FIGURE 2 : VARIATION OF ERROR TERM WITH WEIGHTING FACTOR AT t = 0.1 HOUR

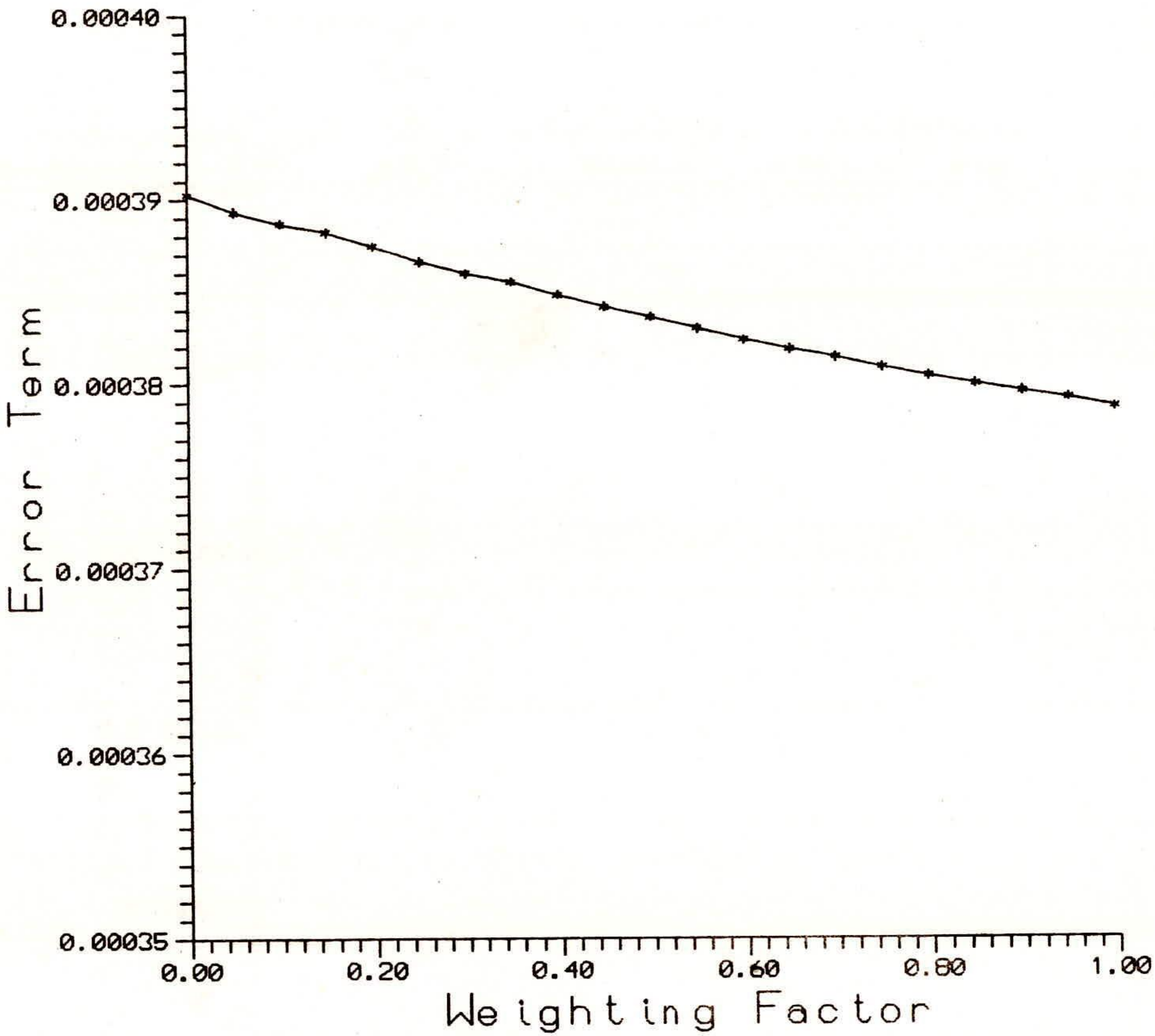


FIGURE 3 : VARIATION OF ERROR TERM WITH WEIGHTING FACTOR AT t = 0.2 HOUR

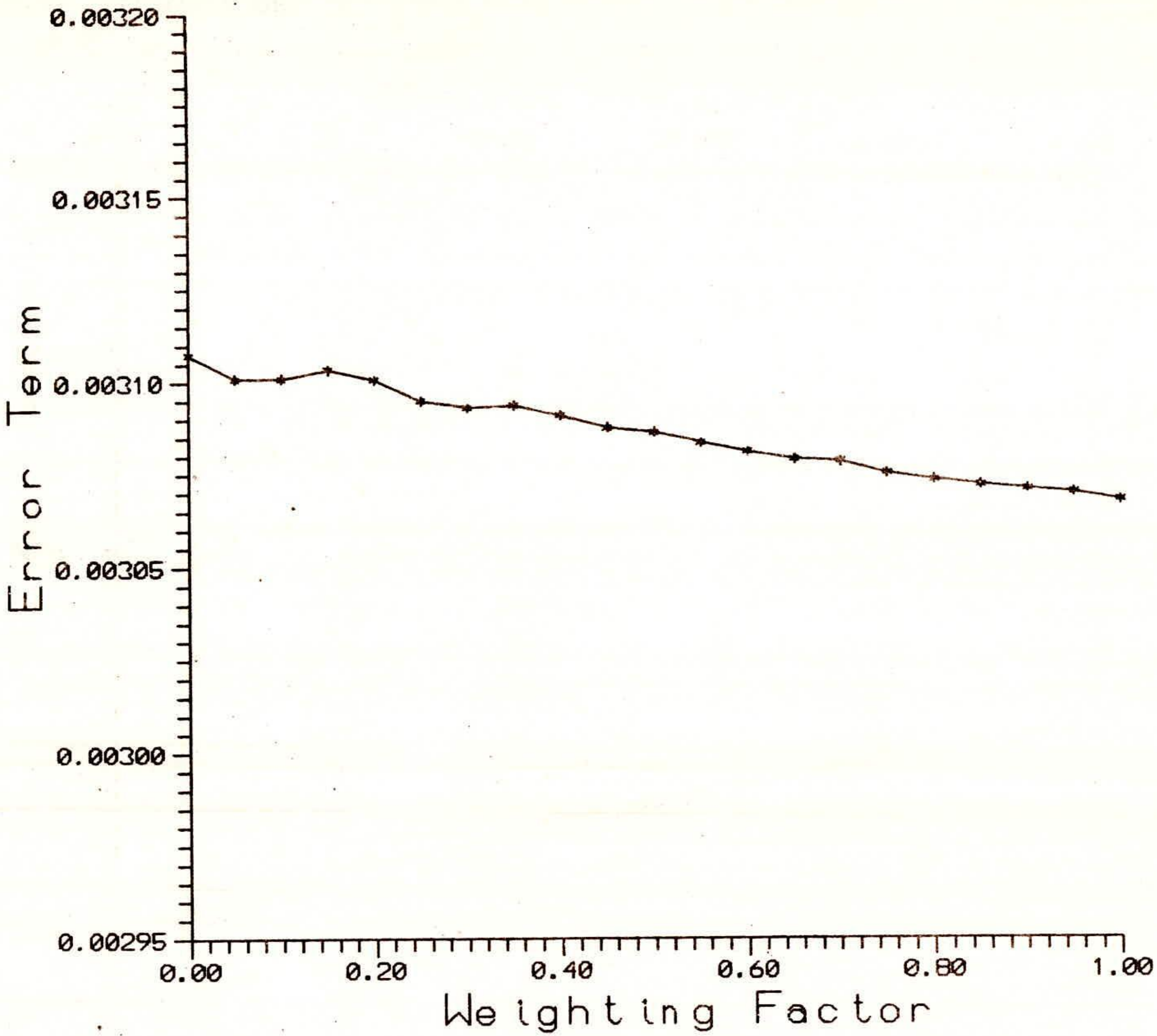


FIGURE 4 : VARIATION OF ERROR TERM WITH WEIGHTING FACTOR AT t = 0.8 HOUR

Tables 3, 4 and 5 present the comparison between water content profiles determined with the solution of Philip and the simulated water content profiles for explicit scheme (a = 0), Crank-Nicolson scheme (a = 0.5) and implicit scheme (a = 1) at t = 0.1 hour, 0.2 hour and 0.8 hour respectively. In all cases, the rate of advance of the water front is particularly well described. Some discrepancies are found between numerical water content profiles and quasi-analytical solution in the low water content domain. However, all the numerical schemes yield comparable results which are not significantly different from the quasi-analytical solution. The input data to the model and output for implicit scheme (a = 1) are given in Appendix - II and Appendix - III respectively.

Table 3 : Comparison between Water Content Profiles at t=0.1 hour

| Depth (z) | Water Content ($\theta$) | | | |
|---|---|---|---|---|
| | Philip | Explicit Scheme (a = 0) | Crank-Nicolson Scheme (a = 0.5) | Implicit Scheme (a = 1) |
| 10 | 0.2484 | 0.247574 | 0.247565 | 0.247550 |
| 11 | 0.2420 | 0.241063 | 0.241051 | 0.241032 |
| 12 | 0.2356 | 0.231798 | 0.231784 | 0.231760 |
| 13 | 0.2217 | 0.218237 | 0.218227 | 0.218201 |
| 14 | 0.2040 | 0.198274 | 0.198285 | 0.198273 |
| 15 | 0.1787 | 0.170837 | 0.170915 | 0.170961 |
| 16 | 0.1491 | 0.140986 | 0.141156 | 0.141294 |
| 17 | 0.1247 | 0.118997 | 0.119176 | 0.119335 |
| 18 | 0.1130 | 0.107607 | 0.107727 | 0.107839 |
| 19 | 0.1054 | 0.102811 | 0.102876 | 0.102937 |

Table 4 : Comparison between Water Content Profiles at t=0.2 hour

| Depth (z) | Water Content ($\theta$) | | | |
|---|---|---|---|---|
| | Philip | Explicit Scheme (a = 0) | Crank-Nicolson Scheme (a = 0.5) | Implicit Scheme (a = 1) |
| 18 | 0.2506 | 0.247249 | 0.247241 | 0.247230 |
| 19 | 0.2451 | 0.242231 | 0.242223 | 0.242209 |
| 20 | 0.2395 | 0.235506 | 0.235496 | 0.235480 |
| 21 | 0.2320 | 0.226299 | 0.226291 | 0.226274 |
| 22 | 0.2201 | 0.213518 | 0.213516 | 0.213502 |
| 23 | 0.2038 | 0.195913 | 0.195931 | 0.195932 |
| 24 | 0.1806 | 0.173123 | 0.173185 | 0.173225 |
| 25 | 0.1567 | 0.148020 | 0.148140 | 0.148237 |
| 26 | 0.1332 | 0.126794 | 0.126937 | 0.127064 |
| 27 | 0.1172 | 0.113176 | 0.113294 | 0.113403 |
| 28 | 0.1109 | 0.106002 | 0.106079 | 0.106152 |
| 29 | 0.1047 | 0.102619 | 0.102664 | 0.102706 |

Table 5 : Comparison between Water Content Profiles at t=0.8 hour

| Depth (z) | Water Content ($\theta$) | | | |
|---|---|---|---|---|
| | Philip | Explicit Scheme (a = 0) | Crank-Nicolson Scheme (a = 0.5) | Implicit Scheme (a = 1) |
| 66 | 0.2490 | 0.246294 | 0.246286 | 0.246276 |
| 67 | 0.2448 | 0.241825 | 0.241816 | 0.241805 |
| 68 | 0.2406 | 0.236003 | 0.235994 | 0.235981 |
| 69 | 0.2364 | 0.228291 | 0.228282 | 0.228270 |
| 70 | 0.2286 | 0.217948 | 0.217944 | 0.217934 |
| 71 | 0.2198 | 0.204080 | 0.204087 | 0.204086 |
| 72 | 0.2063 | 0.186024 | 0.186056 | 0.186077 |
| 73 | 0.1891 | 0.164453 | 0.164525 | 0.164586 |
| 74 | 0.1686 | 0.142607 | 0.142719 | 0.142819 |
| 75 | 0.1482 | 0.124907 | 0.125027 | 0.125139 |
| 76 | 0.1305 | 0.113293 | 0.113390 | 0.113484 |
| 77 | 0.1165 | 0.106719 | 0.106786 | 0.106851 |
| 78 | 0.1072 | 0.103300 | 0.103341 | 0.103382 |

Implicit methods are preferable in view of their stability, even for fairly large time steps thus keeping computer costs low, and their flexibility for solving flow problems when saturated and unsaturated zones have to be considered simultaneously, since for C = 0 one simply has to solve the Laplace's equation.

## 6.0 CONCLUSIONS

A numerical model has been developed for finite difference solution of the non-linear Richards equation describing transient, one-dimensional water flow through the unsaturated porous medium. The solution is applicable to homogeneous and isotropic soils in which the functional relationships between hydraulic conductivity, moisture content and soil moisture tension do not show hysteresis properties.

A modification in the explicit, implicit and Crank-Nicolson schemes has been incorporated by introducing a weighting factor. The simulated water content profiles were compared with those computed through quasi-analytical solution of Philip for the condition of a constant pressure at the soil surface. The implicit scheme was found to give better agreement between the two.

The closer agreement between water content distributions obtained with the model and Philip's quasi-analytical solution indicate that numerical model is a reliable tool for predicting infiltration of water into soil. Considering computer time and stability problems, the implicit finite difference approximation has the widest range of applicability for predicting water movement in soil with both saturated and non-saturated regions.

## REFERENCES


1. Feddes, R. A., P. Kabat, P. J. T. Van Bakel, J. J. B. Bronswijk and J. Halbertsma (1988), "Modelling Soil Water Dynamics in the Unsaturated Zone - State of the Art", Journal of Hydrology, Vol. 100, pp. 69-111.

2. Haverkamp, R. and M. Vauclin (1979), "A Note on Estimating Finite Difference Interbloc Hydraulic Conductivity Values for Transient Unsaturated Flow Problems", Water Resources Research, Volume 15, No. 1, February 1979, pp. 181-187.

3. Haverkamp, R., M. Vauclin, J. Touma, P. J. Wierenga and G. Vachaud (1977), "A Comparison of Numerical Simulation Models for One-Dimensional Infiltration", Soil Sci. Soc. Am. J., Volume 41, pp. 285-294.

4. Kumar, C. P. and G. C. Mishra (1991), "Development of a Soil Moisture Prediction Model", Technical Report No. TR-104, National Institute of Hydrology, Roorkee, 1990-91, 52 pp.

5. Philip, J. R. (1957), "The Theory of Infiltration", Soil Sci., Volume 83 , pp. 345-357 and 435-448.

6. Philip, J. R. (1958), "The Theory of Infiltration 6.", Soil Sci., Volume 85, pp. 278-286.

7. Philip, J. R. (1969), "Theory of Infiltration", Advances in Hydroscience, Volume 5, pp. 215-305, Academic Press, New York.

8. Remson, I., G. M. Hornberger and F. J. Molz (1971), "Numerical Methods in Subsurface Hydrology", Wiley-Interscience, New York, 389 pp.

9. Richards, L. A. (1931), "Capillary Conduction of Liquids through Porous Mediums", Physics, Vol. 1, pp. 318-333.

```
C       SOIL MOISTURE SIMULATION BY IMPROVED NUMERICAL METHOD
C
        DIMENSION SUB(90), SUP(90), DIAG(90), B(90)
        DIMENSION H(90,2), CCC(90,2)
        DIMENSION THETA(90,2), HYDCON(90,2)
        DIMENSION PHIL1(10), PHIL2(12), PHIL3(13)
        OPEN(UNIT=1,FILE='IMPROV.DAT',STATUS='OLD')
        OPEN(UNIT=2,FILE='IMPROV.OUT',STATUS='NEW')
C
C       J REFERS TO TIME
C       I REFERS TO DEPTH
C       Z      = DEPTH (CM), ORIENTED POSITIVELY DOWNWARD
C       A      = WEIGHTAGE ASSIGNED TO THE SPACE DERIVATIVES
C                AT THE (j+1)th TIME LEVEL
C       (1-A)  = WEIGHTAGE ASSIGNED TO THE SPACE DERIVATIVES
C                AT THE jth TIME LEVEL
C       THETA  = VOLUMETRIC MOISTURE CONTENT (CUBIC CM / CUBIC CM)
C       H      = SOIL WATER PRESSURE (RELATIVE TO THE ATMOSPHERE)
C                EXPRESSED IN CM OF WATER
C       THETAR = RESIDUAL MOISTURE CONTENT
C       THETAS = MOISTURE CONTENT AT SATURATION
C       THETAU = MOISTURE CONTENT AT THE SURFACE NODE
C                (UPPER BOUNDARY CONDITION)
C       BETA1, CONA  = PARAMETERS IN THE HYDRAULIC CONDUCTIVITY
C                      AND SOIL WATER PRESSURE RELATIONSHIP
C       BETA2, ALPHA = PARAMETERS IN THE MOISTURE CONTENT AND
C                      SOIL WATER PRESSURE RELATIONSHIP
C       HYDCON = HYDRAULIC CONDUCTIVITY OF THE SOIL (CM/HOUR)
C       AKS    = HYDRAULIC CONDUCTIVITY AT SATURATION (CM/HOUR)
C       DELT   = TIME STEP (HOURS)
C       DELZ   = DEPTH INTERVAL (CM)
C       NTIME  = NUMBER OF TIME STEPS
C       NNODE  = NUMBER OF NODES
C       CCC    = SPECIFIC WATER CAPACITY (/CM) defined as d(theta)/dh
C
        READ(1,11)A
11      FORMAT(F12.3)
        READ(1,12)THETAR,THETAS,THETAU
12      FORMAT(3F12.3)
        READ(1,13)BETA1,BETA2
13      FORMAT(2F12.3)
        READ(1,14)CONA,ALPHA
14      FORMAT(2F12.3)
        READ(1,15)AKS
15      FORMAT(F12.3)
        READ(1,16)DELT,DELZ
16      FORMAT(F12.8,F12.3)
        READ(1,17)NTIME,NNODE
17      FORMAT(I4,8X,I2)
C
```

```
C       READING OF INITIAL CONDITIONS
C
        READ(1,18)(THETA(I,1),I=1,NNODE)
18      FORMAT(5F12.6)
C
C       READING OF PHILIP'S QUASI-ANALYTICAL SOLUTION RESULTS
C
        READ(1,*)
        READ(1,19)(PHIL1(M),M=1,10)
        READ(1,*)
        READ(1,19)(PHIL2(M),M=1,12)
        READ(1,*)
        READ(1,19)(PHIL3(M),M=1,13)
19      FORMAT(5F12.4)
C
        WRITE(2,21)
21      FORMAT(2X,'SOIL MOISTURE SIMULATION BY IMPROVED NUMERICAL METHOD')
        WRITE(2,22)
22      FORMAT(/2X,'RICHARDS EQUATION SOLVED IN TERMS OF H')
        WRITE(2,23)A
23      FORMAT(/2X,'A = ',F5.3)
        WRITE(2,24)
24      FORMAT(/2X,'THETAR',9X,'THETAS',9X,'THETAU')
        WRITE(2,25)THETAR,THETAS,THETAU
25      FORMAT(2X,F5.3,10X,F5.3,10X,F5.3)
        WRITE(2,26)
26      FORMAT(/2X,'BETA1',10X,'BETA2')
        WRITE(2,27)BETA1,BETA2
27      FORMAT(2X,F5.3,10X,F5.3)
        WRITE(2,28)
28      FORMAT(/2X,'CONA',11X,'ALPHA')
        WRITE(2,29)CONA,ALPHA
29      FORMAT(2X,F11.3,4X,F11.3)
        WRITE(2,30)
30      FORMAT(/2X,'AKS')
        WRITE(2,31)AKS
31      FORMAT(2X,F6.3)
        WRITE(2,32)
32      FORMAT(/2X,'DELT',11X,'DELZ')
        WRITE(2,33)DELT,DELZ
33      FORMAT(2X,F9.8,6X,F5.3)
        WRITE(2,34)
34      FORMAT(/2X,'NTIME',10X,'NNODE')
        WRITE(2,35)NTIME,NNODE
35      FORMAT(I6,11X,I3)
        WRITE(2,36)
36      FORMAT(/2X,'SOIL MOISTURE AT DIFFERENT NODES')
        WRITE(2,37)
37      FORMAT(/2X,'INITIAL CONDITIONS'/)
        WRITE(2,38)(THETA(I,1),I=1,NNODE)
38      FORMAT(5F12.6)
        WRITE(2,*)
C
```

```
      DO 100 I=1,NNODE
      H(I,1)=-(ALPHA*(THETAS-THETA(I,1))/(THETA(I,1)
     1        -THETAR))**(1./BETA2)
100   CONTINUE
C
C     GENERATION OF UPPER BOUNDARY CONDITION
C
      THETA(1,1)=THETAU
      THETA(1,2)=THETA(1,1)
      H(1,1)=-(ALPHA*(THETAS-THETA(1,1))/(THETA(1,1)
     1        -THETAR))**(1./BETA2)
      H(1,2)=H(1,1)
C
C     GENERATION OF LOWER BOUNDARY CONDITION
C
      THETA(NNODE,2)=THETA(NNODE,1)
      H(NNODE,2)=H(NNODE,1)
C
C     FORMULATION OF NUMERICAL SCHEME
C
      E1=BETA1/BETA2
      E2=(THETAS-THETAR)
      E3=ALPHA**E1
      E4=CONA*AKS
      E5=1./BETA2*ALPHA**(1./BETA2)
C
      DO 200 I = 2, NNODE-1
      THETA(I,2) = THETA(I,1)
      H(I,2) = H(I,1)
200   CONTINUE
C
      DO 300 J = 2, NTIME
C
      ITER = 1
400   CONTINUE
C
      DO 500 I = 1, NNODE
      HYDCON(I,1) = E4/(CONA+(ABS(H(I,1)))**BETA1)
      CCC(I,1) = 1./(E5*E2)*(THETAS-THETA(I,1))**(-1./BETA2+1.)*
     1           (THETA(I,1)-THETAR)**(1./BETA2+1.)
      HYDCON(I,2) = E4/(CONA+(ABS(H(I,2)))**BETA1)
      CCC(I,2) = 1./(E5*E2)*(THETAS-THETA(I,2))**(-1./BETA2+1.)*
     1           (THETA(I,2)-THETAR)**(1./BETA2+1.)
500   CONTINUE
C
      DO 600 I = 2, NNODE-1
      F1 = (1-A)*CCC(I,1)+A*CCC(I,2)
      F2 = (1-A)*((HYDCON(I,1)*HYDCON(I+1,1))**0.5)
     1     +A*((HYDCON(I,2)*HYDCON(I+1,2))**0.5)
      F3 = (1-A)*((HYDCON(I-1,1)*HYDCON(I,1))**0.5)
     1     +A*((HYDCON(I-1,2)*HYDCON(I,2))**0.5)
      DIAG(I-1) = F1+A*(F2+F3)*DELT/DELZ**2
      SUB(I-1)  = -A*F3*DELT/DELZ**2
      SUP(I-1)  = -A*F2*DELT/DELZ**2
```

```
      B(I-1)     = ((1-A)*F3*DELT/DELZ**2)*H(I-1,1)
     1             +(F1-(1-A)*(F2+F3)*DELT/DELZ**2)*H(I,1)
     2             +((1-A)*F2*DELT/DELZ**2)*H(I+1,1)+(F3-F2)*DELT/DELZ
600   CONTINUE
C
      B(1)=B(1)-SUB(1)*H(1,2)
      B(NNODE-2)=B(NNODE-2)-SUP(NNODE-2)*H(NNODE,2)
      DO 700 I = 1, NNODE-3
700   SUB(I)=SUB(I+1)
      M=NNODE-2
      CALL TRID(M,SUP,SUB,DIAG,B)
C
      SUM = 0
      DO 800 I = 1, NNODE-2
      SUM = SUM + (H(I+1,2)-B(I))**2
800   CONTINUE
C
      DO 900 I = 1, NNODE-2
      H(I+1,2)=B(I)
900   CONTINUE
C
      IF (SUM.LE.0.0001) GO TO 1000
      ITER = ITER + 1
      GO TO 400
1000  CONTINUE
C
      DO 1100 I = 2, NNODE-1
      THETA(I,2)=ALPHA*(THETAS-THETAR)/(ALPHA+ABS(H(I,2))**BETA2)
     1           +THETAR
1100  CONTINUE
C
      IF (J.EQ.901)  GO TO 50
      IF (J.EQ.1801) GO TO 60
      IF (J.EQ.7201) GO TO 70
      GO TO 80
C
50    CONTINUE
      SUM1 = 0
      DO 102 N = 1, 10
      ERR1 = (THETA(N+10,2)-PHIL1(N))**2
      SUM1 = SUM1 + ERR1
102   CONTINUE
      WRITE(2,103)A,SUM1
103   FORMAT(//2X,'A =   ', F5.3,11X,'ERROR TERM = ',F12.9)
      GO TO 40
C
60    CONTINUE
      SUM2 = 0
      DO 104 N = 1, 12
      ERR2 = (THETA(N+18,2)-PHIL2(N))**2
      SUM2 = SUM2 + ERR2
104   CONTINUE
      WRITE(2,105)A,SUM2
105   FORMAT(//2X,'A =   ', F5.3,11X,'ERROR TERM = ',F12.9)
```

```
      GO TO 40
C
70    CONTINUE
      SUM3 = 0
      DO 106 N = 1, 13
      ERR3 = (THETA(N+66,2)-PHIL3(N))**2
      SUM3 = SUM3 + ERR3
106   CONTINUE
      WRITE(2,107)A,SUM3
107   FORMAT(//2X,'A =   ', F5.3,11X,'ERROR TERM = ',F12.9)
C
40    CONTINUE
      ITIME=J-1
      HOUR=ITIME*DELT
      WRITE(2,108)ITER
108   FORMAT(/2X,'ITERATION = ',I5)
      WRITE(2,109)ITIME,HOUR
109   FORMAT(/2X,'TIME STEP = ',I5,6X,'DURATION = ',F12.6,2X,'HOUR'/)
      WRITE(2,110)(THETA(I,2),I=1,NNODE)
110   FORMAT(5F12.6)
80    CONTINUE
C
      DO 90 I = 2, NNODE-1
      THETA(I,1)=THETA(I,2)
      H(I,1)=H(I,2)
90    CONTINUE
C
300   CONTINUE
      STOP
      END
C
      SUBROUTINE TRID(M,SUP,SUB,DIAG,B)
      DIMENSION SUP(90),SUB(90),DIAG(90),B(90)
      N=M
      NN=N-1
      SUP(1)=SUP(1)/DIAG(1)
      B(1)=B(1)/DIAG(1)
      DO 111 I = 2, N
      II=I-1
      DIAG(I)=DIAG(I)-SUP(II)*SUB(II)
      IF (I.EQ.N) GO TO 111
      SUP(I)=SUP(I)/DIAG(I)
111   B(I)=(B(I)-SUB(II)*B(II))/DIAG(I)
      DO 222 K = 1, NN
      I=N-K
222   B(I)=B(I)-SUP(I)*B(I+1)
      RETURN
      END
```

| | | | | |
|---|---|---|---|---|
| 1.000 | | | | |
| 0.075 | 0.287 | 0.267 | | |
| 4.740 | 3.960 | | | |
| 1175000.000 | 1611000.000 | | | |
| 34.000 | | | | |
| 0.00011111 | 1.000 | | | |
| 7201 | 90 | | | |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |
| 0.100000 | 0.100000 | 0.100000 | 0.100000 | 0.100000 |

| | | | | |
|---|---|---|---|---|
| 0.2484 | 0.2420 | 0.2356 | 0.2217 | 0.2040 |
| 0.1787 | 0.1491 | 0.1247 | 0.1130 | 0.1054 |

| | | | | |
|---|---|---|---|---|
| 0.2506 | 0.2451 | 0.2395 | 0.2320 | 0.2201 |
| 0.2038 | 0.1806 | 0.1567 | 0.1332 | 0.1172 |
| 0.1109 | 0.1047 | | | |

| | | | | |
|---|---|---|---|---|
| 0.2490 | 0.2448 | 0.2406 | 0.2364 | 0.2286 |
| 0.2198 | 0.2063 | 0.1891 | 0.1686 | 0.1482 |
| 0.1305 | 0.1165 | 0.1072 | | |

SOIL MOISTURE SIMULATION BY IMPROVED NUMERICAL METHOD

RICHARDS EQUATION SOLVED IN TERMS OF H

A = 1.000

| THETAR | THETAS | THETAU |
|---|---|---|
| .075 | .287 | .267 |

| BETA1 | BETA2 |
|---|---|
| 4.740 | 3.960 |

| CONA | ALPHA |
|---|---|
| 1175000.000 | 1611000.000 |

| AKS |
|---|
| 34.000 |

| DELT | DELZ |
|---|---|
| .00011111 | 1.000 |

| NTIME | NNODE |
|---|---|
| 7201 | 90 |

SOIL MOISTURE AT DIFFERENT NODES

INITIAL CONDITIONS

| | | | | |
|---|---|---|---|---|
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |

A = 1.000 ERROR TERM = .000243760

ITERATION = 2

TIME STEP = 900 DURATION = .099999 HOUR

| | | | | |
|---|---|---|---|---|
| .267000 | .266377 | .265630 | .264728 | .263631 |
| .262284 | .260611 | .258504 | .255806 | .252278 |
| .247550 | .241032 | .231760 | .218201 | .198273 |
| .170961 | .141294 | .119335 | .107839 | .102937 |
| .101050 | .100363 | .100122 | .100040 | .100013 |
| .100004 | .100001 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |

A = 1.000 ERROR TERM = .000378676

ITERATION = 2

TIME STEP = 1800 DURATION = .199998 HOUR

| | | | | |
|---|---|---|---|---|
| .267000 | .266817 | .266604 | .266358 | .266071 |
| .265738 | .265349 | .264896 | .264364 | .263740 |
| .263001 | .262125 | .261076 | .259813 | .258276 |
| .256386 | .254033 | .251057 | .247230 | .242209 |
| .235480 | .226274 | .213502 | .195932 | .173225 |
| .148237 | .127064 | .113403 | .106152 | .102706 |
| .101161 | .100489 | .100203 | .100083 | .100033 |
| .100013 | .100005 | .100002 | .100001 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |
| .100000 | .100000 | .100000 | .100000 | .100000 |

A = 1.000 ERROR TERM = .003068143

ITERATION = 2

TIME STEP = 7200 DURATION = .799992 HOUR

| | | | | |
|---|---|---|---|---|
| .267000 | .266999 | .266999 | .266998 | .266997 |
| .266996 | .266995 | .266994 | .266993 | .266991 |
| .266989 | .266987 | .266985 | .266982 | .266979 |
| .266976 | .266972 | .266968 | .266963 | .266958 |
| .266952 | .266945 | .266937 | .266929 | .266919 |
| .266908 | .266895 | .266881 | .266866 | .266848 |
| .266828 | .266805 | .266780 | .266751 | .266719 |
| .266682 | .266641 | .266595 | .266543 | .266483 |
| .266417 | .266341 | .266256 | .266159 | .266050 |
| .265926 | .265785 | .265625 | .265443 | .265236 |
| .264999 | .264729 | .264419 | .264063 | .263653 |
| .263180 | .262630 | .261990 | .261242 | .260361 |
| .259318 | .258075 | .256579 | .254764 | .252536 |
| .249767 | .246276 | .241805 | .235981 | .228270 |
| .217934 | .204086 | .186077 | .164586 | .142819 |
| .125139 | .113484 | .106851 | .103382 | .101645 |
| .100794 | .100382 | .100183 | .100087 | .100041 |
| .100019 | .100009 | .100004 | .100001 | .100000 |

| | | |
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