THEORY

Nonleaky Aquifer

The partial differential equation describing unsteady radially symmetric flow in a nonleaky homogeneous confined aquifer of constant thickness can be written as

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t} \qquad (1)$$

in which h is the piezometric head at a radius r from the pumped well at a time t since the start of pumping. T and S are the transmissivity and storage coefficient, respectively. Equation (1) assumes that the pumping well fully penetrates the aquifer layer during the test. A solution to equation (1) which satisfies the continuity requirement and the conditions

$$h(r, 0) = h_o$$

$$h(\infty, t) = h_o$$

as given by Theis (1935) is

$$h_o - h = \frac{Q_w}{4\pi T} \int_{u=r^2S/4Tt}^{\infty} (e^{-u}/u)\, du \qquad (2)$$

where h_o is the initial piezometric head, and Q_w is the constant rate of pumping.

Equation (2) is usually written as

$$s = \frac{Q_w}{4\pi T} \times W(u) \qquad (3)$$

where s is the drawdown ($= h_o - h$) and W(u) is the Theis well function representing the exponential integral of equation (2).

Equation (3) is nonlinear in T and S. These parameters can be estimated by nonlinear regression analysis. The algorithm used for determining T and S is the one due to Marquardt (1963) as explained below. Let s_i^* be the drawdown at any instant computed by substituting the initial trial values of parameters T^* and S^* in equation (3). Let ΔT and ΔS be the respective increments in T^* and S^* to yield the improved estimates T and S at the end of each trial and let s_i be the corresponding drawdown value given as,

$$s_i = f(T, S)$$

Expanding s_i by Taylor series about the trial values,

$$s_i = f(T^* + \Delta T, S^* + \Delta S)$$

$$\text{or} \quad s_i = f(T^*, S^*) + \frac{\partial f(T^*, S^*)}{\partial T^*}\Delta T + \frac{\partial f(T^*, S^*)}{\Delta S^*}\Delta S$$

$$\text{or} \quad s_i = s_i^* + \frac{\partial s_i^*}{\partial T^*}\Delta T + \frac{\partial s_i^*}{\partial S^*}\Delta S \qquad (4)$$

The increments ΔT and ΔS are determined such that the sum of squares of the difference between the observed and the calculated drawdowns is minimum,

$$\text{i.e.,} \quad \underset{\Delta T, \Delta S}{\text{Min}} \ \text{sum} = \sum_{i=1}^{N} (s_i^0 - s_i)^2 \qquad (5)$$

where s_i^0 is the observed drawdown at any instant.

The linearized model given by equation (4) is substituted in equation (5) and normal equations are formed by setting the partial derivatives of the objective function given by equation (5) with respect to T and S equal to zero

$$\text{i.e.,} \quad \frac{\partial\, \text{sum}}{\partial T^*} = 0$$

$$\text{and} \quad \frac{\partial\, \text{sum}}{\partial S^*} = 0$$

These equations will take the form

$$(A^T A)\,\underline{\Delta A} = A^T (s^o - s^*) \qquad (6)$$

$$\text{where} \quad A = \begin{bmatrix} \frac{\partial s_1^*}{\partial T^*} & \frac{\partial s_1^*}{\partial S^*} \\ \frac{\partial s_2^*}{\partial T^*} & \frac{\partial s_2^*}{\partial S^*} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{\partial s_N^*}{\partial T^*} & \frac{\partial s_N^*}{\partial S^*} \end{bmatrix} ; \quad \underline{\Delta A} = \begin{bmatrix} \Delta T \\ \cdot \\ \Delta S \end{bmatrix}$$

A^T is the transpose of the matrix A; s^o and s^* are the vectors of observed and calculated drawdowns, respectively.

The normal equations (6) are solved for $\underline{\Delta A}$ and the new drawdowns are calculated by substituting the improved estimates (T, S) of the parameters in equation (3). The error criterion is checked and if the same is not satisfied, the process is repeated with the updated estimates of the parameters.

In order to ensure convergence with relatively poor starting values, equation (6) is modified as

$$(A^T A + \lambda I)\,\underline{\Delta A} = A^T (s^o - s^*) \qquad (7)$$

where λ is the convergence factor and I is the identity matrix. Initial values of λ are large and decrease towards zero as convergence is reached.

Lecture delivered by Mr.S.K. Singh, Scientist, NIH

DETERMINATION OF AQUIFER PARAMETERS USING MARQUARDT ALGORITHM

Type of aquifer : Confined
Description of the pumping well : Well with negligible storage
Description of the observation well : Well with neglible storage

Definition of Variables

- QP - Pumping rate (cubic meter/unit time), unit time is HCF of NTIMES(NTV) (see below),
- R1 - Distance of the observation well from the pumping well,
- NPUMPT - Duration of pumping,
- NTV - No of observations,
- NTIMES - Times at which observations are taken, NTV values to be given,
- ODRW - Observed drawdown, NTV values to be given.

PREPARATION OF INPUT DATA

File name : MC1.DAT

```
Line          1 : Initial guess of T and PHI
Line          2 : Lower Limit of the parameters T and PHI
Line          3 : Upper Limit of the parameters T and PHI
Line          4 : QP, R1, NPUMPT, NTV
Line    5 -   n : (NTIMES(I),I=1,NTV)
Line n+1 - end  : (ODRW(I),I=1,NTV)
```

Note : Transmissivity is determined per unit time.

A well was pumped for 7000 min at the rate of 18.9236 $m^3/10$ min. The drawdown observations were taken at a well at a distance 99.9 m. The unit time was 10 min. Determine T & PHI.

Sample File MC1.DAT

```
 0.40     0.001700
 0.01     0.00001
10.95     0.500
18.9236   99.9    700    34
1      2      3      4      5      7      10     12     15     18
20     26     30     40     50     60     70     80     90     100
120    150    170    200    250    300    360    400    450    500
550    600    650    700
0.219  0.357  0.447  0.511  0.564  0.643  0.729  0.780  0.838  0.878
0.907  0.979  1.009  1.090  1.163  1.217  1.256  1.289  1.313  1.341
1.402  1.450  1.474  1.532  1.597  1.635  1.687  1.717  1.742  1.782
1.797  1.804  1.837  1.860
```

Running program MADCAT1
The user should create the input files MAR.DAT in the directory

in which the program MADCAT1 resides. The program is invoked by typing MADCAT1 at the system prompt. The Result File MC1.OUT obtained for the above sample is given below.

RESULT FILE MC1.OUT

```
 *** MARQUARDT ALGORITHM ***
Iter No   1  Obj Fun =  .1394E+04  T & PHI =      400000    .001700
Iter No   2  Obj Fun =  .2937E+03  T & PHI =      786359    .002742
Iter No   3  Obj Fun =  .5346E+02  T & PHI =    1.511954    .003507
Iter No   4  Obj Fun =  .8357E+01  T & PHI =    2.735118    .002794
Iter No   5  Obj Fun =  .1685E+01  T & PHI =    4.280962    .001171
Iter No   6  Obj Fun =  .7661E-01  T & PHI =    5.336930    .000879
Iter No   7  Obj Fun =  .2429E-02  T & PHI =    5.634017    .000830
Iter No   8  Obj Fun =  .2208E-02  T & PHI =    5.651053    .000828
Iter No   9  Obj Fun =  .2208E-02  T & PHI =    5.651106    .000828

 *** SOLUTION OF THE PROBLEM ***
 T   =   5.65111
 PHI =   0.00083
```

Therefore, T = 5.65 sq m/10 min
PHI = 0.00083

***

MADCAT2

DETERMINATION OF AQUIFER PARAMETERS USING MARQUARDT ALGORITHM
(WITH RECOVERY)

Type of aquifer : Confined
Description of the pumping well : Well with negligible storage
Description of the observation well : Well with neglible storage

INPUT DATA

Definition of Variables

QP - Pumping rate (cubic meter/unit time), unit time is HCF of NTIMES(NTV) (see below),
R1 - Distance of the observation well from the pumping well,
NPUMPT - Duration of pumping,
NTV - No of observations,
NTIMES - Times at which observations are taken, NTV values to be given,
ODRW - Observed drawdown, NTV values to be given.

SAMPLE INPUT

A well was pumped for 7000 min at the rate of 18.9236 $m^3/10$ min. The drawdown observations were taken at a well at a distance 99.9 m up to 14000 min. The unit time was 10 min. Determine aquifer parameters T & PHI.

File name : MC2.DAT

```
Line          1 : Initial guess of T and PHI
Line          2 : Lower Limit of the parameters T and PHI
Line          3 : Upper Limit of the parameters T and PHI
Line          4 : QP, R1, NPUMPT, NTV
Line    5 -   n : (NTIMES(I),I=1,NTV)
Line n+1 - end : (ODRW(I),I=1,NTV)
```

Note : Transmissivity is determined per unit time.

Sample File MC2.DAT

```
 0.10     0.01700
 0.01     0.0001
 10.00    0.5000

 18.9236   99.9   700   55
    1      2      3      4      5      7     10     12     15     18
   20     26     30     40     50     60     70     80     90    100
  120    150    170    200    250    300    360    400    450    500
  550    600    650    700
  701    702    703    704    706    708    710    714    720    726
  730    740    750    770    800    840    900    960   1050   1167
 1400
```

```
0.219  0.357  0.447  0.511  0.564  0.643  0.729  0.780  0.838  0.878
0.907  0.979  1.009  1.090  1.163  1.217  1.256  1.289  1.313  1.341
1.402  1.450  1.474  1.532  1.597  1.635  1.687  1.717  1.742  1.782
1.797  1.804  1.837  1.860
1.650  1.512  1.427  1.326  1.266  1.196  1.137  1.051  0.958  0.891
0.850  0.780  0.719  0.642  0.557  0.476  0.377  0.320  0.242  0.179
0.104
```

****

Running program MADCAT2

The user should create the input files MC2.DAT in the directory in which the program MADCAT2 resides. The program is invoked by typing MADCAT2 at the system prompt. The Result file MC2.OUT created for the above sample data is given below.

RESULT FILE MC2.OUT

```
        Determination of Aquifer parameters Using
                  MARQUARDT ALGORITHM

 Iter No   1  Obj Fun =  .9956E+03  T & PHI =   .100000   .017000
 Iter No   2  Obj Fun =  .2308E+03  T & PHI =   .231797   .032107
 Iter No   3  Obj Fun =  .6225E+02  T & PHI =   .598453   .054302
 Iter No   4  Obj Fun =  .2864E+02  T & PHI =  1.749724   .057216
 Iter No   5  Obj Fun =  .2292E+02  T & PHI =  2.318044   .041598
 Iter No   6  Obj Fun =  .1689E+02  T & PHI =  2.790527   .025285
 Iter No   7  Obj Fun =  .1144E+02  T & PHI =  3.197116   .014365
 Iter No   8  Obj Fun =  .7344E+01  T & PHI =  3.560820   .008396
 Iter No   9  Obj Fun =  .4597E+01  T & PHI =  3.887761   .005292
 Iter No  10  Obj Fun =  .2858E+01  T & PHI =  4.177920   .003634
 Iter No  11  Obj Fun =  .1785E+01  T & PHI =  4.430376   .002696
 Iter No  12  Obj Fun =  .6466E+00  T & PHI =  5.213781   .000733
 Iter No  13  Obj Fun =  .2236E-01  T & PHI =  5.617170   .000818
 Iter No  14  Obj Fun =  .1832E-01  T & PHI =  5.659461   .000823
 Iter No  15  Obj Fun =  .1832E-01  T & PHI =  5.659837   .000823
 Iter No  16  Obj Fun =  .1832E-01  T & PHI =  5.659837   .000823

  *** SOLUTION OF THE PROBLEM ***

 T   =   5.65984
 PHI =    .00082
```

Therefore, T = 5.66 sq m/10 min
PHI = 0.00082

***

MADCAT3

Determination of Aquifer Parameters Using Pumping Data of a Large Diameter Well

Type of aquifer : Confined
Description of the pumping well : Large Diameter Well.
Description of the observation well : Large Diameter Well.

DEFINITION OF VARIABLES

RW1	:	Radius of the pumping well,
RW2	:	Radius of the observation well,
RC1	:	Radius of the casing of the pumping well,
RC2	:	Radius of the casing of the observation well,
R1	:	Distance of the observation well from the pumping well,
QP	:	Rate of pumping per unit time,
NTIME	:	No of time steps upto the end of which observations are made,
MPUMPT	:	No of time steps upto the end of which pumping is carried out,
INDEX	:	Its value is 1 if the drawdown observations are made at the pumping well and 2 if they are made at an observation well at a distance R1,
ODRW	:	Observed_drawdown.

A well was pumped for 7000 min at the rate of 18.9236 $m^3/10$ min. The drawdown observations were taken at a well at a distance 99.9 m. The unit time was 10 min. Determine T & PHI.

Sample File MC3.DAT

```
0.10   0.1700
0.01   0.0001
1.95   0.500

0.1   0.1   3.0   0.1   10.   0.50   0.30   0.001
240   120   1

0.0175 0.0348 0.0518 0.0688 0.0855 0.1021 0.1185 0.1347 0.1508 0.1667
0.1825 0.1981 0.2136 0.2289 0.2441 0.2592 0.2741 0.2889 0.3035 0.3180
0.3324 0.3466 0.3607 0.3747 0.3885 0.4023 0.4159 0.4294 0.4427 0.4560
0.4691 0.4821 0.4950 0.5078 0.5205 0.5331 0.5455 0.5579 0.5701 0.5822
0.5943 0.6062 0.6180 0.6298 0.6414 0.6529 0.6644 0.6757 0.6869 0.6981
0.7091 0.7201 0.7310 0.7417 0.7524 0.7630 0.7735 0.7839 0.7943 0.8045
0.8147 0.8248 0.8348 0.8447 0.8545 0.8643 0.8740 0.8836 0.8931 0.9025
0.9119 0.9212 0.9304 0.9395 0.9486 0.9576 0.9665 0.9753 0.9841 0.9928
1.0015 1.0100 1.0185 1.0270 1.0353 1.0436 1.0519 1.0600 1.0681 1.0762
1.0842 1.0921 1.0999 1.1077 1.1155 1.1231 1.1307 1.1383 1.1458 1.1532
1.1606 1.1679 1.1752 1.1824 1.1895 1.1966 1.2037 1.2107 1.2176 1.2245
1.2313 1.2381 1.2448 1.2515 1.2581 1.2647 1.2712 1.2777 1.2841 1.2905
1.2793 1.2683 1.2574 1.2467 1.2361 1.2256 1.2153 1.2050 1.1949 1.1849
```

1.1750	1.1652	1.1555	1.1459	1.1364	1.1270	1.1177	1.1085	1.0994	1.0904
1.0815	1.0727	1.0639	1.0553	1.0467	1.0382	1.0298	1.0215	1.0133	1.0052
0.9971	0.9891	0.9812	0.9734	0.9656	0.9579	0.9503	0.9428	0.9353	0.9279
0.9206	0.9133	0.9061	0.8990	0.8920	0.8850	0.8781	0.8712	0.8644	0.8577
0.8510	0.8444	0.8379	0.8314	0.8250	0.8186	0.8123	0.8061	0.7999	0.7938
0.7877	0.7817	0.7757	0.7698	0.7639	0.7581	0.7524	0.7467	0.7410	0.7354
0.7299	0.7244	0.7189	0.7135	0.7081	0.7028	0.6976	0.6924	0.6872	0.6821
0.6770	0.6720	0.6670	0.6620	0.6571	0.6523	0.6475	0.6427	0.6380	0.6333
0.6286	0.6240	0.6194	0.6149	0.6104	0.6060	0.6015	0.5972	0.5928	0.5885
0.5843	0.5800	0.5759	0.5717	0.5676	0.5635	0.5594	0.5554	0.5514	0.5475
0.5436	0.5397	0.5359	0.5320	0.5283	0.5245	0.5208	0.5171	0.5134	0.5098

****

Running program MADCAT3

The user should create the input files MC3.DAT in the directory in which the program MADCAT3 resides. The program is invoked by typing MADCAT3 at the system prompt. The Result file RC3.OUT created for the above sample data is given below.

RESULT FILE MC3.OUT

```
Determination of Aquifer parameters Using
          MARQUARDT ALGORITHM

Iter No   1  Obj Fun = 0.2062E+02   T & PHI =     0.100000    0.170000
Iter No   2  Obj Fun = 0.1111E+01   T & PHI =     0.145392    0.500000
Iter No   3  Obj Fun = 0.1041E+00   T & PHI =     0.175533    0.277894
Iter No   4  Obj Fun = 0.5197E-01   T & PHI =     0.180384    0.282560
Iter No   5  Obj Fun = 0.4849E-01   T & PHI =     0.182329    0.254188
Iter No   6  Obj Fun = 0.4549E-01   T & PHI =     0.185062    0.229482
Iter No   7  Obj Fun = 0.4273E-01   T & PHI =     0.187230    0.207648
          ---- Intermediate Lines Deleted ----
Iter No 100  Obj Fun = 0.2564E-06   T & PHI =     0.299789    0.001010
Iter No 101  Obj Fun = 0.2446E-06   T & PHI =     0.299311    0.001009
Iter No 102  Obj Fun = 0.2347E-06   T & PHI =     0.299330    0.001008

 *** SOLUTION OF THE PROBLEM ***
T   =    0.29983
PHI =    0.00101
```

Therefore, T = 0.30 sq m/min
PHI = 0.00101

***