The differential equation which governs an axially symmetric radial unsteady ground water flow in unconfined aquifer with delayed yield is (Boulton, 1954)

$$T\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right) = \phi \frac{\partial s}{\partial t} + \alpha\phi_y \int_0^\infty \frac{\partial s}{\partial c} e^{-\alpha(t-c)} dc \qquad ...(2.2)$$

The solution of Eq.(2.2) for constant pumping rate given by Boulton (1963) is

$$s = \frac{Q}{4\pi T}\int_0^\infty \frac{2}{x}\left[1 - e^{-\mu_1}\left(\text{Cosh}\ \mu_2 + \frac{\alpha t\eta(1-x^2)}{2\mu_2}\text{Sinh}\ \mu_2\right)\right]J_0\left(\frac{rx}{\gamma D}\right)dx$$
$$...(2.3)$$

Where

$\eta \quad = \quad 1 + \frac{\phi_y}{\phi}$,

ϕ_y = total volume of delayed yield from storage per unit drawdown per unit horizontal area which is commonly referred as specific yield,

ϕ = volume of water instantaneously released from storage per unit drawdown per unit horizontal area which is the effective early time storage coefficient,

$\gamma = \sqrt{\left(\frac{\eta-1}{\eta}\right)}$,

$\frac{1}{\alpha}$ = Boulton's delay index,

$D = \sqrt{T/(\alpha\phi_y)}$,

$\mu_1 = \frac{\alpha t\eta(1+x^2)}{2}$,

$\mu_2 = \frac{\alpha t\sqrt{\eta^2(1+x^2)^2 - 4\eta x^2}}{2}$,

$J_0(\ )$ = Bessel function of first kind, zero order, The solution given at Eq.(2.3) has been obtained by Boulton with the assumption (besides other usual assumptions) that the drawdown is very small in comparison to the thickness of the aquifer. Eq.(2.2) being linear, method of superposition and proportionality are applicable. If Q = 1.0 and pumping continues indefinitely, Eq.(2.3) gives the response of a linear system due to unit step excitation. Designating K(m) as the unit step kernel (response due to an unit step excitation), which is the drawdown at the end of time step m due to continuous pumping at unit quantity per unit time period, the discrete kernel coefficients $\partial(m)$ can be expressed as

$$\partial(m) = K(m) - K(m-1) \qquad \ldots(3.1)$$

Substituting m for t in Eq.(2.3) and replacing Cosh μ_2 and Sinh μ_2 by $(e^{\mu_2} + e^{-\mu_2})/2$ and $(e^{\mu_2} - e^{-\mu_2})/2$ respectively and rearranging, the unit step kernel is written as

$$K(m) = \frac{1}{4\pi T} \int_0^\infty \frac{2}{x} [1 - \frac{1}{2} (e^{-(\mu_1-\mu_2)}(1 + \frac{\alpha m\eta(1-x^2)}{2\mu_2}) + e^{-(\mu_1+\mu_2)}(1 - \frac{\alpha m\eta(1-x^2)}{2\mu_2}))] J_0(\frac{r x}{\gamma D}) dx \qquad \ldots(3.2)$$

The integral appearing in Eq.(3.2) is an improper integral as one of the limits of integration is infinite. For finite values of η the numerical integration of the improper integral takes considerable computer time to obtain results of reasonable accuracy. The following is an efficient method for evaluation of K(m) for any value of η. For given values of aquifer parameters it is found that the limit of the term

$$[1 - \frac{1}{2}(e^{-(\mu_1-\mu_2)}(1 + \frac{\alpha m\eta(1-x^2)}{2\mu_2}) + e^{-(\mu_1+\mu_2)}(1 - \frac{\alpha m\eta(1-x^2)}{2\mu_2}))]$$

in Eq.(3.2) tends to 1 as the dummy variable x increases.

Let, beyond $x=x_1$ this term has a value equal to $1-\epsilon$, where ϵ is as small as .000001.

Eq.(3.2) can be written as

$$\begin{aligned} K(m) = {} & \frac{1}{4\pi T}\int_0^{x_1}\frac{2}{x}[1 - \frac{1}{2}(e^{-(\mu_1-\mu_2)}(1 + \frac{\alpha m\eta(1 - x^2)}{2\mu_2}) \\ & + e^{-(\mu_1+\mu_2)}(1 - \frac{\alpha m\eta(1-x^2)}{2\mu_2}))]\, J_0(\frac{r\,x}{\gamma D})\, dx \\ & + \frac{1}{4\pi T}\int_{x_1}^{\infty}(1-\epsilon)\,\frac{2}{x}\, J_0(\frac{rx}{\gamma D})\, dx \qquad \ldots(3.3) \\ = {} & I_1 + I_2 \qquad \ldots(3.4) \end{aligned}$$

For evaluation of the proper integral I_1, numerical integration is carried out assuming dx = .001. This value of dx has been adopted after studying the effect of dx on the accuracy of the results.

The integration

$I_2 = \int_{x_1}^{\infty}\frac{2}{x}\, J_0(\frac{rx}{\gamma D})\, dx$ is carried out as follows :

Let

$$y = \frac{r}{\gamma D}\, x$$

Then

$$I_2 = \int_{\frac{rx_1}{\gamma D}}^{\infty} \frac{2}{y} J_o(y)\, dy \qquad \ldots(3.5)$$

Depending upon the numerical values of $\frac{r}{\gamma D} x_1$ the following approximations can be used for evaluation of the improper integral I_2.

For $\frac{r}{\gamma D} x_1 < 2$ (Abramowitz and Stegun 1970, pp.481)

$$\int_{\frac{rx_1}{\gamma D}}^{\infty} \frac{J_o(y)}{y}\, dy = -\, 0.5772156 - \log_e\left(\frac{rx_1}{2\gamma D}\right) - \sum_{p=1}^{\infty} \frac{(-1)^p \left(\frac{rx_1}{2\gamma D}\right)^{2p}}{2p(p!)^2} \qquad \ldots(3.6)$$

The series appearing in Eq.(3.6) is a rapidly converging one.

For $5 \leq \frac{rx_1}{\gamma D} \leq \infty$ (Abramowitz and Stegun 1970,pp.482)

$$\int_{\frac{rx_1}{\gamma D}}^{\infty} \frac{J_o(y)}{y}\, dy$$

$$= \frac{2g_1\left(\frac{rx_1}{\gamma D}\right) J_o\left(\frac{rx_1}{\gamma D}\right)}{\left(\frac{rx_1}{\gamma D}\right)^2} - \frac{g_o\left(\frac{rx_1}{\gamma D}\right) J_1\left(\frac{rx_1}{\gamma D}\right)}{\left(\frac{rx_1}{\gamma D}\right)} \qquad \ldots(3.7)$$

Where $J_o(\ )$ and $J_1(\ )$ are Bessel functions of first kind of zero and first order respectively ;

$$g_o\left(\frac{rx_1}{\gamma D}\right) = \sum_{p=0}^{9} (-1)^p\, a_p \left(\frac{rx_1}{5\gamma D}\right)^{-2p} + \left(\frac{rx_1}{\gamma D}\right) ;$$

and

$$g_1\left(\frac{rx_1}{\gamma D}\right) = \sum_{p=0}^{9} (-1)^p\, b_p \left(\frac{rx_1}{5\gamma D}\right)^{-2p} + \left(\frac{rx_1}{\gamma D}\right) .$$

$$\left|\epsilon\left(\frac{rx_1}{\gamma D}\right)\right| \leq 2 \times 10^{-7}$$

The values of a_p and b_p are as follows :

p	a_p	b_p
0	1.0	1.0
1	0.159992815	0.319985629
2	0.101619385	0.304858155
3	0.130811585	0.523246341
4	0.207404022	1.037020112
5	0.283300508	1.699803050
6	0.279029488	1.953206413
7	0.178915710	1.431325684
8	0.066228328	0.596054956
9	0.010702234	0.107022336

For $2 < \frac{rx_1}{\gamma D} < 5$,

the integral $\int_{x_1}^{\infty} \frac{2}{x} J_o\left(\frac{rx}{\gamma D}\right) dx$ is evaluated in the following manner :

$$\int_{x_1}^{\infty} \frac{2}{x} J_o\left(\frac{rx}{\gamma D}\right)dx = \int_{x_1}^{x_2} \frac{2}{x} J_o\left(\frac{rx}{\gamma D}\right)dx + \int_{x_2}^{\infty} \frac{2}{x} J_o\left(\frac{rx}{\gamma D}\right)dx \quad ...(3.8)$$

Evaluation of

$\int_{x_1}^{x_2} \frac{2}{x} J_o\left(\frac{rx}{\gamma D}\right)dx$ is done numerically and

$\int_{x_2}^{\infty} \frac{2}{x} J_o\left(\frac{rx}{\gamma D}\right) dx$ is done using Eq.(3.7)

because value of x_2 is such that $\frac{rx_2}{\gamma D} \geq 5$.

RESULTS AND DISCUSSION

Let,

$$\text{TERM} = [1 - \tfrac{1}{2}(e^{-(\mu_1-\mu_2)}(1 + \frac{\alpha m\eta(1-x^2)}{2\mu_2})$$
$$+ e^{-(\mu_1+\mu_2)}(1 - \frac{\alpha m\eta(1-x^2)}{2\mu_2}))].$$

The value of TERM at different values of x are presented in Table (3.1) for aquifer parameters : $T = 350\ m^2/day$, $\phi = .003$, $\phi_y = .1$, $\alpha = 13.8/day$. As seen from the table, beyond $x = 0.999$, $TERM = 1 - \epsilon$ and $\epsilon < 0.000001$. Hence when $x > 0.999$ the value of TERM can be taken as $1.x_1$ for the above set of aquifer parameters is therefore equal to 0.999.

Discrete kernel coefficients are generated for the following sets of aquifer parameters

T m^2/day	ϕ	ϕ_y	α 1/day	η
350.0	0.001	.03	20.0	31.0
700.0	0.001	0.03	20.0	31.0

Discrete kernel coefficients are generated when excitation and observation points are different. The generated discrete kernel coefficients are presented in Figs. 3.1 and 3.2. In Table (3.2) discrete kernel coefficients for drawdown in unconfined aquifers without and with delayed yield characteristics having the following parameters: $T = 700.0\ m^2/day$, $\phi = 0.031$; and $T = 700.0\ m^2/day$, $\phi = 0.001$, $\phi_y = .03$, $\alpha = 20.0/day$ respectively have been presented for the purpose of comparison.

The procedure described here can also be extended to evaluate the discrete kernel coefficients when the excitation and response points are same. Fig. 3.3 shows a square grid from which unit quantity of water is withdrawn during the first unit time period (and pumping stopped). In order to find the response at the centre of the grid due to the pulse excitation the grid is divided into 36 equal units as shown. It is envisaged that 36 wells are operating one at a time at the centre of each unit. Using method of superposition the drawdown at the centre of grid when all the 36 wells are operating simultaneously is obtained. Sum of the drawdowns is divided by 36 to arrive at the response due to unit withdrawl from the grid. The discrete kernel coefficient generated is designated as $\delta_{rr}(m)$. The $\delta_{rr}(m)$ values have been plotted in Fig. 3.4.

Using the present procedure the well function $W(u_{ay}, \frac{r}{D})$,[$W(u_{ay}, \frac{r}{D})$ is the well function of an unconfined aquifer having delayed yield characteristics], has been evaluated for $\eta = 10.0$, $\frac{r}{D} = 2.0$ for different values of u_a and u_y ($u_a = \frac{r^2\phi}{4Tt}$, $u_y = \frac{r^2\phi y}{4Tt}$) and the same has been plotted in Fig.3.5. Also, the results obtained by Boulton (1964) for these aquifer paramters have been plotted in the same figure.

In order to compare the well function for finite and infinite values of η, the results obtained by Boulton (1963) for a large value of η ($\eta > 100$) have also been presented in Fig.3.5. It may be seen that the type curve for $\eta > 100$ deviates appreciably from the curve for η=10.0.

CONCLUSIONS

a) An efficient method to evaluate type curves for drawdown in an unconfined aquifer with delayed yield for finite value of η has been described.

b) The discrete kernel coefficients for drawdown in an unconfined aquifer with delayed yield have been obtained.

Table 3.1 Values of 'TERM' for different values of x

x	TERM
$.9900001\times10^{-1}$	.1267259
.1990000	.4213664
.2990000	.7086827
.3989998	.8882457
.4989996	.9671634
.5989998	.9925374
.6990000	.9986624
.7990002	.9998028
.8990004	.9999737
.9990006	.9999961
$.1099001\times10$	.9999992
$.1199001\times10$	.9999997
$.1299001\times10$	.9999999
$.1399001\times10$	.9999999
$.1499002\times10$	.9999999
$.1599002\times10$	1.0000000
$.1699002\times10$	1.0000000

Table 3.2 Discrete kernel coefficients for drawdown in an unconfined aquifer

Time in days	δ_{rp} (with delayed yield) $m/(m^3/day)$ *		δ_{rp} (with out delayed yield) $m/(m^3/day)$ **	
	r = 300m	r = 600m	r = 300m	r = 600m
1	$.2711 \times 10^{-4}$	$.8445 \times 10^{-6}$	$.2510 \times 10^{-4}$	$.4372 \times 10^{-6}$
2	$.3763 \times 10^{-4}$	$.5498 \times 10^{-5}$	$.3879 \times 10^{-4}$	$.5177 \times 10^{-5}$
3	$.3023 \times 10^{-4}$	$.9030 \times 10^{-5}$	$.3064 \times 10^{-4}$	$.9120 \times 10^{-5}$
4	$.2434 \times 10^{-4}$	$.1022 \times 10^{-4}$	$.2451 \times 10^{-4}$	$.1036 \times 10^{-4}$
5	$.2021 \times 10^{-4}$	$.1029 \times 10^{-4}$	$.2029 \times 10^{-4}$	$.1040 \times 10^{-4}$
6	$.1722 \times 10^{-4}$	$.9925 \times 10^{-5}$	$.1728 \times 10^{-4}$	$.1001 \times 10^{-4}$
7	$.1500 \times 10^{-4}$	$.9415 \times 10^{-5}$	$.1502 \times 10^{-4}$	$.9472 \times 10^{-5}$
8	$.1326 \times 10^{-4}$	$.8861 \times 10^{-5}$	$.1329 \times 10^{-4}$	$.8910 \times 10^{-5}$
9	$.1189 \times 10^{-4}$	$.8335 \times 10^{-5}$	$.1191 \times 10^{-4}$	$.8370 \times 10^{-5}$
10	$.1077 \times 10^{-4}$	$.7843 \times 10^{-5}$	$.1078 \times 10^{-4}$	$.7870 \times 10^{-5}$
11	$.9841 \times 10^{-5}$	$.7385 \times 10^{-5}$	$.9853 \times 10^{-5}$	$.7409 \times 10^{-5}$
12	$.9062 \times 10^{-5}$	$.6973 \times 10^{-5}$	$.9070 \times 10^{-5}$	$.6992 \times 10^{-5}$

* T = 700.0 m^2/day, ϕ = 0.001, ϕ_y = .03, α = 20.0/day

**T = 700.0 m^2/day, ϕ = .031.

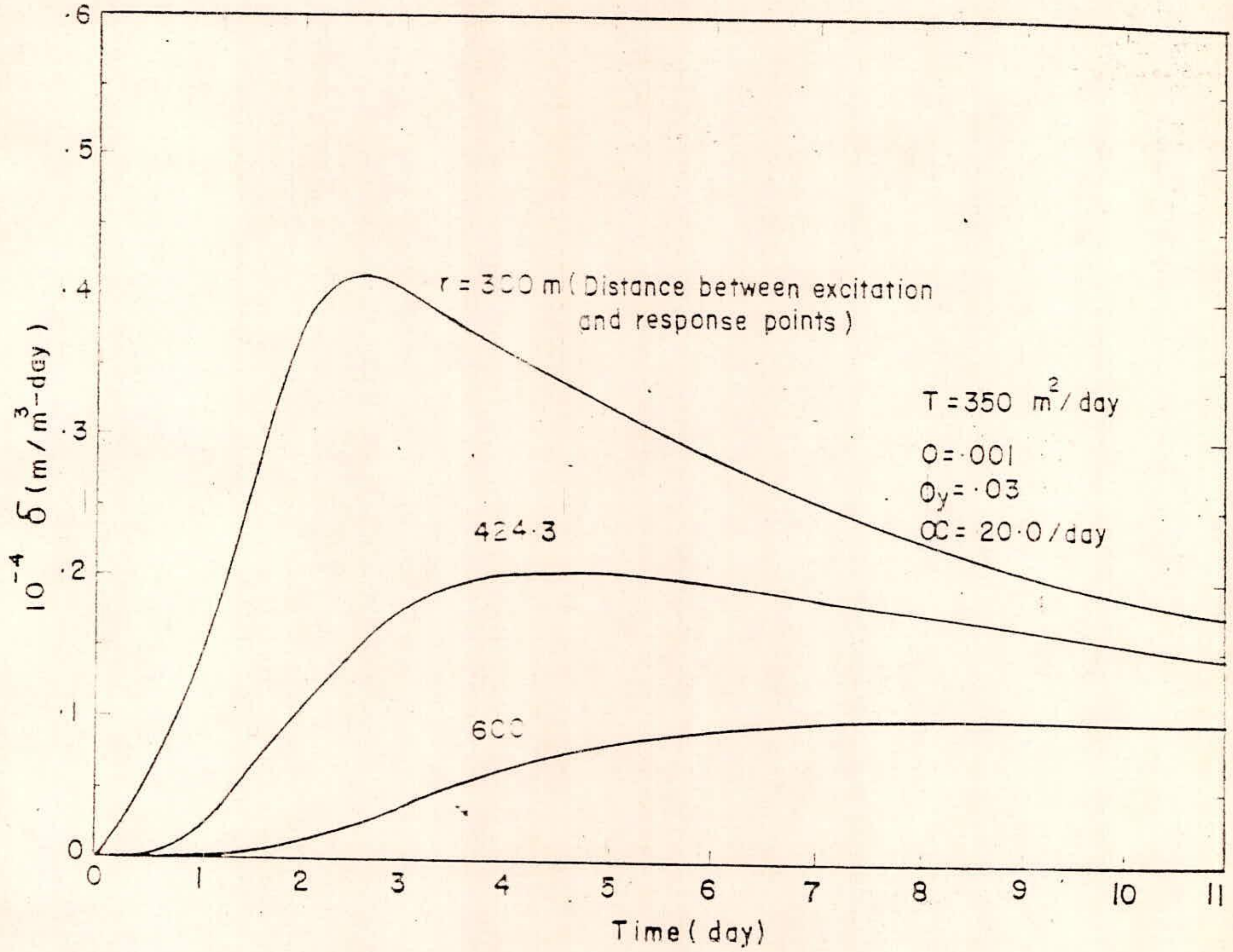

Fig. 3·1 Discrete kernel coefficients for drawdown in an unconfined aquifer having delayed yield ; excitation and response points are different.

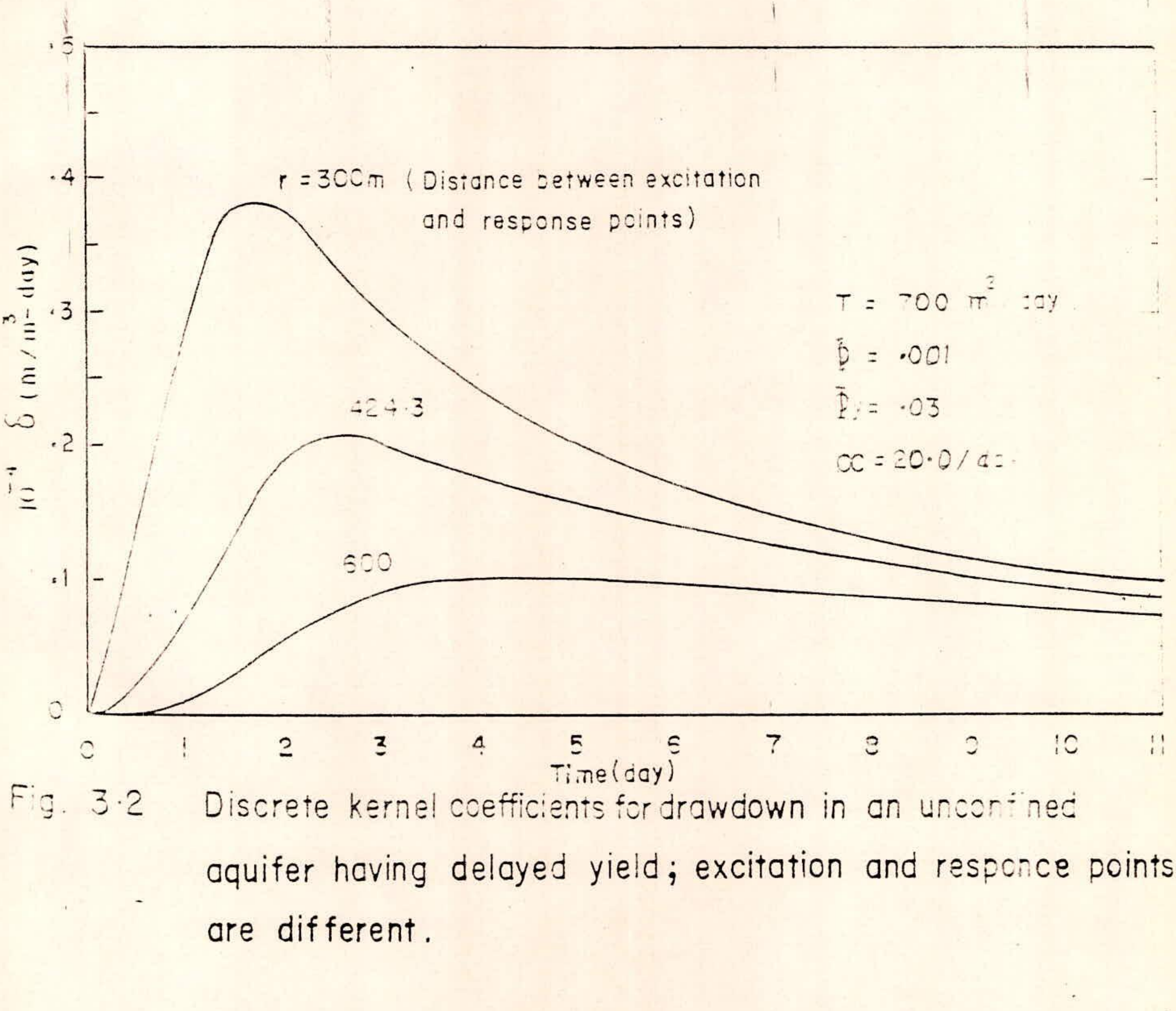

Fig. 3·2 Discrete kernel coefficients for drawdown in an unconfined aquifer having delayed yield; excitation and responce points are different.

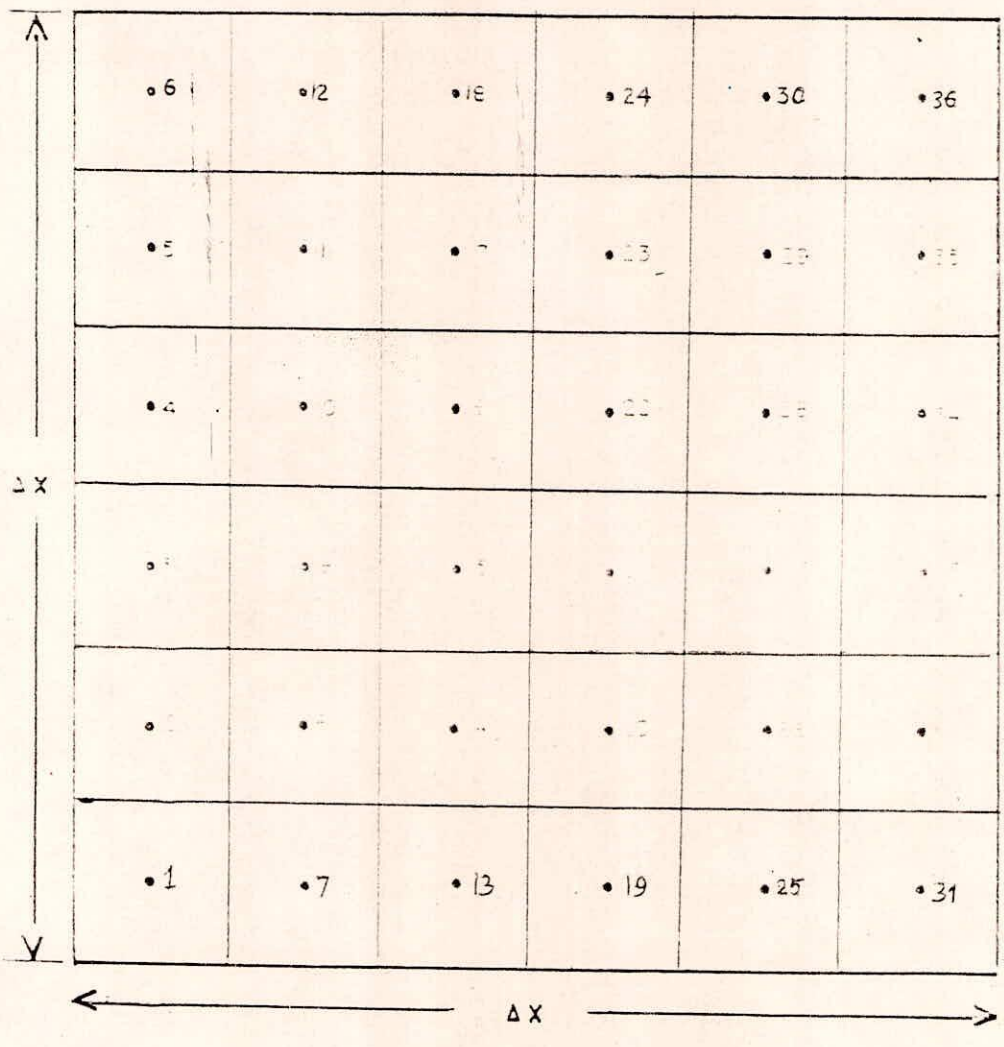

Fig. 3·3 Division of a grid into 36 units for evaluation of response when the excitation and observation points are same.

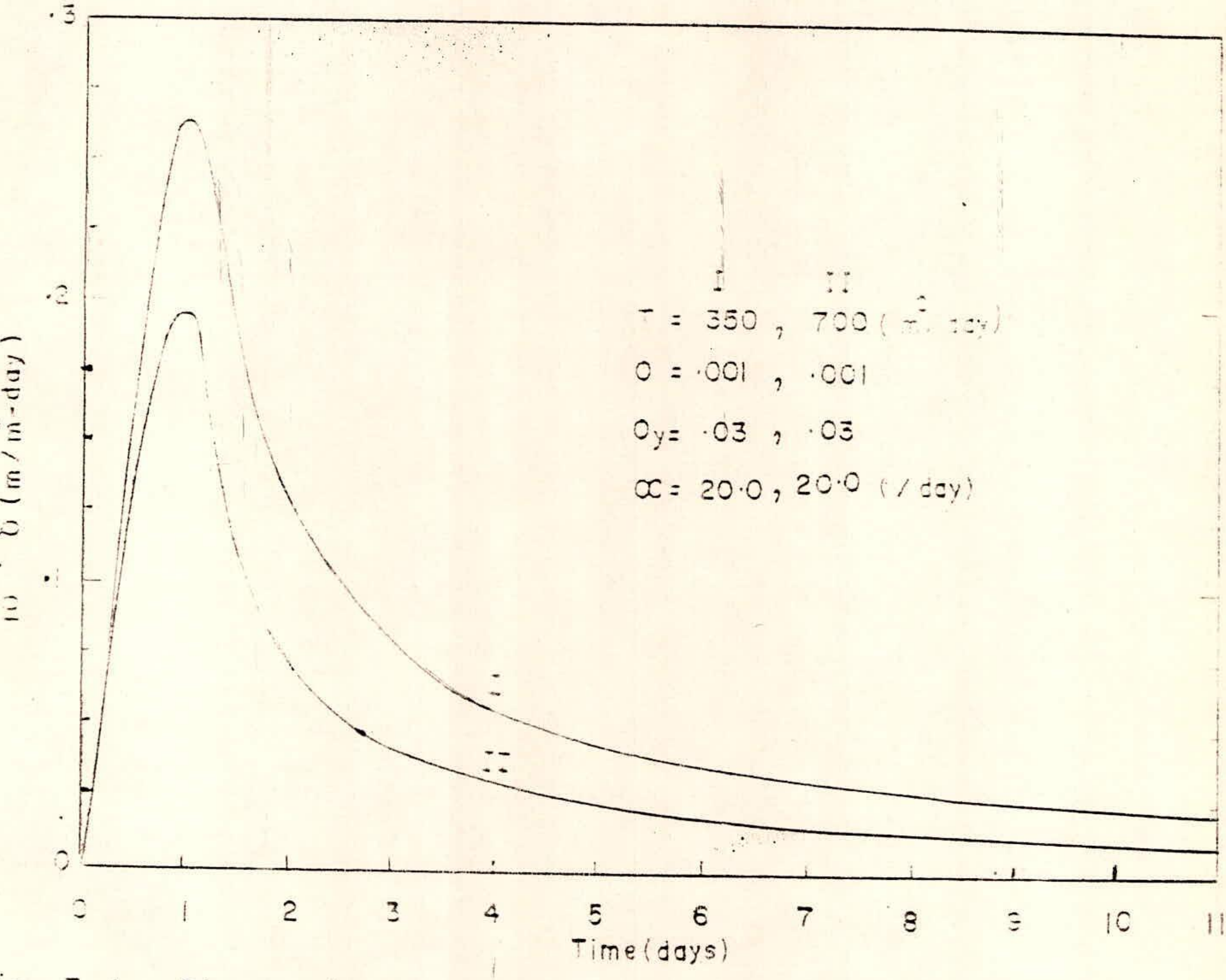

Fig. 3·4 Discrete kernel coefficients for drawdown in an unconfined aquifer having delayed yield ; excitation and response points are same

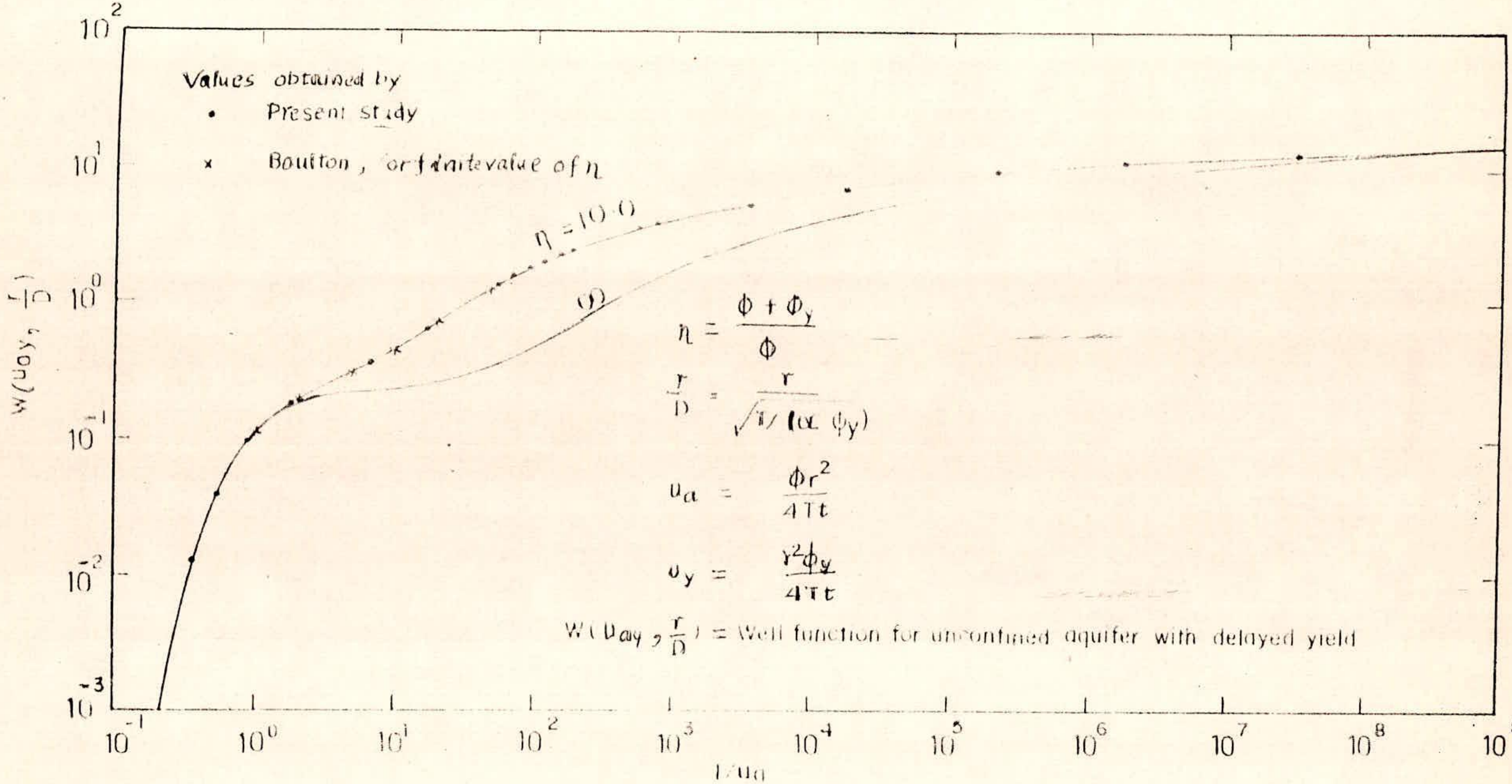

Fig. 3·5 Type curve for an unconfined aquifer with delayed yield for $\frac{r}{D} = 2\cdot0$ and $\eta = 10\cdot0$