NATIONAL INSTITUTE OF HYDROLOGY
ROORKEE

WORKSHOP

ON
GROUND WATER MODELLING - TYSON-WEBER MODEL
(18 - 22 Nov.1985)

LECTURE	:	III	
TOPIC	:	COMPUTER BASED MODELS	
BY	:	DR G.C.MISHRA	
DATE AND TIME	:	19.11.85	10.00 A.M. - 11.00 A.M.
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COMPUTER BASED MODELS

Groundwater hydrology is a quantitative science and mathematics is one of its principal dialects. It will be unwise to ignore the powerful tools of the groundwater trade that rest on an understanding of mathematics. The mathematical methods upon which classical studies of groundwater flow are based were borrowed by the early researchers from areas of applied mathematics originally developed for the treatment of problems of heat flow, electricity and magnetism. With the advent of the digital computer, many of the important recent advances in the analysis of groundwater system have been based on much different mathematical approaches generally known as numerical methods.

To fully define a transient boundary value problem for subsurface flow, one needs to know

i) the size and shape of the region of flow,

ii) the equation of flow within the region,

iii) the boundary conditions around the boundaries of the region,

iv) the initial conditions in the region,

v) the spatial distribution of the hydrogeologic parameters that control the flow, and

vi) a mathematical method of solution.

If the boundary - value problem is for a steady - state system, requirement (iv) is removed.

Consider the simple groundwater flow problem illustrated in Figure 1.

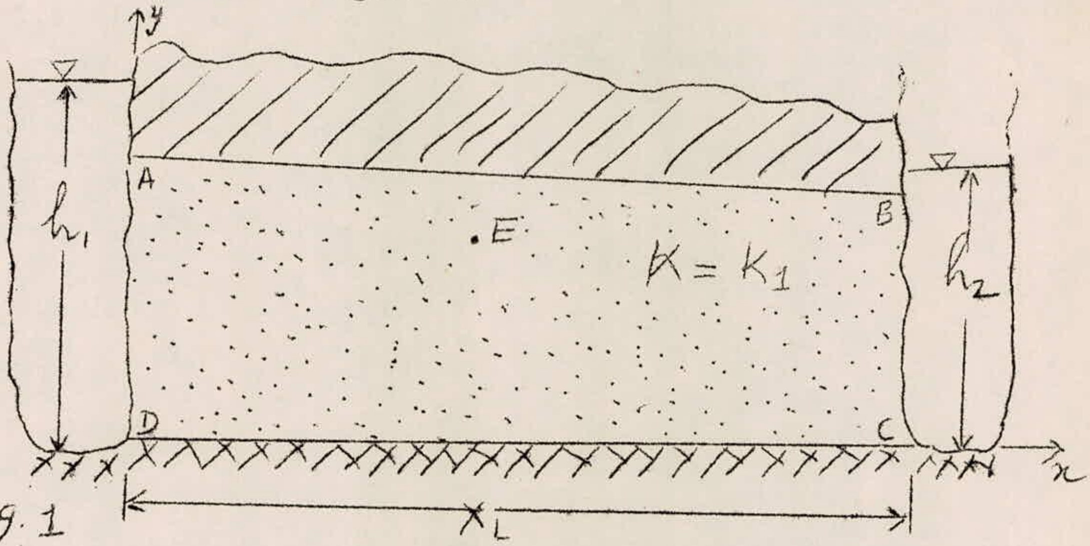


Fig. 1

The region ABCD contains a homogeneous, isotropic porous medium of hydraulic conductivity K_1. The boundaries AB and CD are impermeable. The hydraulic heads on AD and BC are h_1 and h_2 respectively. Assuming steady flow and setting h_1 = 100m and h_2 = 0m, the hydraulic head at point 'E' will be 50m. Apparently the implicit use of properties (i), (iii) and (v) have been made and the solution has been arrived at by inspection. If it is required to know the distribution of hydraulic head in a complex flow domain that has been shown in Figure 2, the equation of flow and a mathematical method of solution are needed.

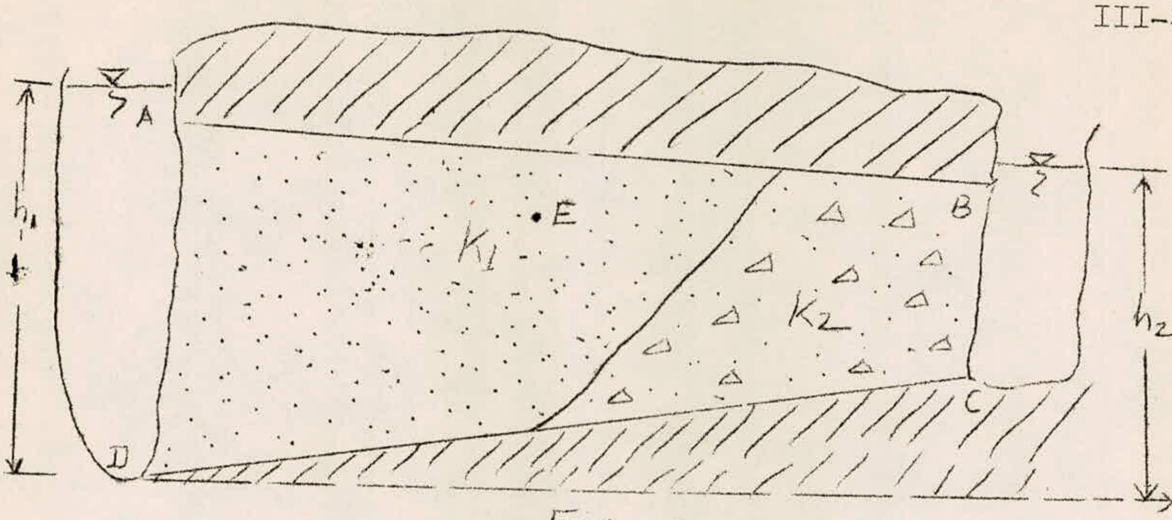


Fig-2

The method of solution can be categorized into five approaches :

1. Solution by inspection
2. Solution by grap hical techniques
3. Solution by analog model
4. Solution by analytical mathematical technique
5. Solution by numerical mathematical technique

Tractable mathematical solution to boundary value problems can be obtained provided the medium is homogeneous and the boundary conditions are idealized. The following two boundary val ue problems have been discussed as examplies: consider a confined aquifer bounded by a fully penetrating straight river reach on one side and extending to infinity on the other side. The aquifer is of uniform thickness and is homogeneous.(Fig. 3).

Let the river - aqu ifer system is initially at rest condition i.e. the water level in the river and the water table position in the aquifer at all places are same.

Let the river stage attains a new position instantaneously, and let the drawdown in the river stage is σ. Let the river stage after attaining the new position remains uncharged indefinitely. For such an idealized flow domain the governing differential equation is

$$\frac{\partial s}{\partial t} = \frac{\phi}{T} \frac{\partial^2 s}{\partial x^2} \qquad \ldots.(1)$$

in which

s = drawdown measured from the datum coinciding with the initial water table position,

T = transmissivity,

ϕ = storage coefficient.

The solution to the linear partial differential equation for the initial condition $s(x,o) = o$, and boundary conditions $s(o,t) = \sigma$ is given by

$$S(x,t) = [\, 1 - \text{erf}\, (\frac{x}{\sqrt{4\beta t}})\,] \qquad \ldots..(2)$$

$$\text{where erf}\,(Y) = \frac{2}{\sqrt{\pi}} \int_0^Y e^{-z^2}\, dz$$

and $\beta = \frac{T}{\phi}$.

The flow at any section is given by the expression

$$Q(x,t) = - \frac{\sigma T}{\sqrt{\pi \beta t}} \qquad \ldots..(3)$$

If the boundary condition changes with time, solution for drawdown and flow for the variable boundary condition can be arrived making use of equation 2 and 3 as follows :

Let $K(t) = 1 - \mathrm{erf}\left(\frac{x}{\sqrt{4\beta t}}\right)$, ...(4)

$$S(x,t) = \int_0^t \frac{\partial\sigma(z)}{\partial z} K(t-z)\, dz \quad .....(5)$$

$$S(x,t) = \int_0^t \frac{\partial\sigma(z)}{\partial z}\left[1 - \mathrm{erf}\left(\frac{x}{\sqrt{4\beta(t-z)}}\right)\right]dz. \quad (6)$$

and

$$Q(x,t) = \int_0^t - \frac{\partial\sigma(z)}{\partial z} \frac{T}{\sqrt{\pi\beta(t-z)}}\, dz \quad .....(7)$$

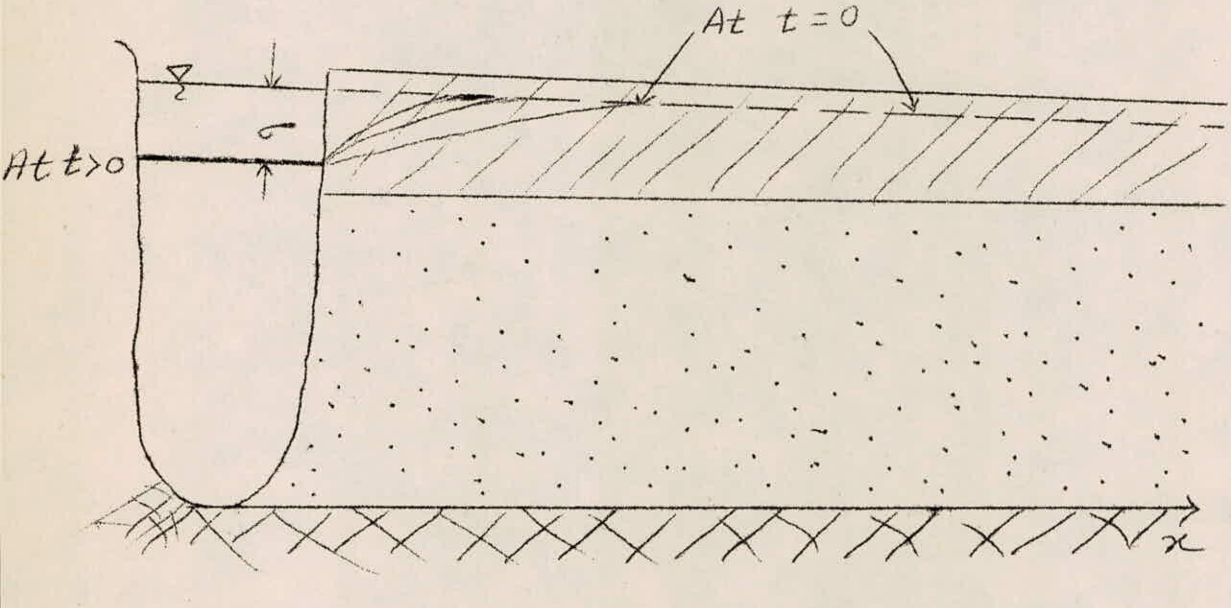
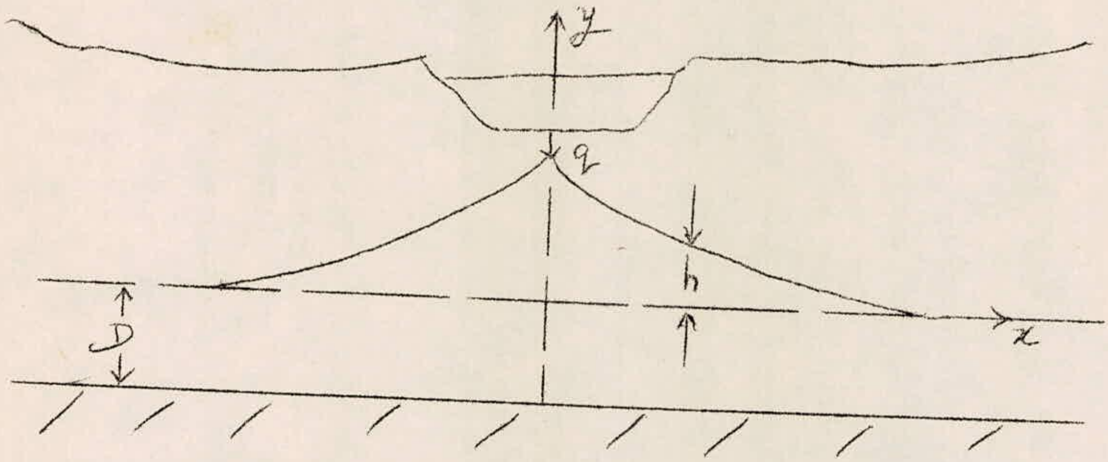


Fig - 3

Let us consider a 2nd example in which a prescribed flow rate is the boundary condition at one end and at other end a fixed head boundary condition exists.



Assuming that the Dupuit Forchheimer assumptions are valid, the two dimensional problem in x,y plane can be regarded as a one dimensional flow problem. The governing differential equation will be

$$\frac{\partial^2 h}{\partial x^2} = \frac{T}{\phi}\frac{\partial h}{\partial t}$$

This equation is to be solved subject to the boundary conditions

$$-KD\ \frac{\partial h}{\partial x}(o,t) = \frac{q}{2}$$

$$h(\infty,t) = o,$$

and the initial condition

$$h(x,o) = o.$$

The solution is given by (Glover,1977)

$$h = \frac{q\sqrt{4\pi\beta t}}{2\pi KD}\left(\frac{x}{\sqrt{4\beta t}}\right)\int_{\frac{x}{\sqrt{4\beta t}}}^{\infty}\frac{e^{-u^2}}{u^2}\,du$$

which simplifies to

$$h = \frac{q\sqrt{4\pi\beta t}}{2\pi KD} \left(\frac{x}{\sqrt{4\pi\beta t}}\right) \sqrt{\pi} \left[\frac{e^{-\frac{x^2}{4\beta t}}}{\frac{x}{\sqrt{4\beta t}}\sqrt{\pi}} - 1 + \text{erf}\left(\frac{x}{\sqrt{4\beta t}}\right)\right]$$

The flow rate at any section is given by

$$q_x = \frac{q}{2}\left[1 - \text{erf}\left(\frac{x}{\sqrt{4\pi\beta t}}\right)\right]$$

The error function can be evaluated using the following rational approximation (Abramowitz and Stegun, 1959).

$$\text{erf}(x) = 1 - (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) e^{-x^2} + e(x)$$

$p = .3275911$, $a_1 = .254829592$

$a_2 = -.284496736$ $a_3 = 1.421413741$

$a_4 = -1.453152027$ $a_5 = 1.061405429$

In case of linear system the time parameter can be conveniently discretised and within each time step the imput to the system can be assumed to be constant but it can vary from time step to timestep. Knowing the response of the system for a unit puls excitation, solution to the boundary value problem can be conveniently obtained. The problem becomes simple when the flow domain is homogneous. This method is recognised as discrete kernel method and tractable solutions for may complex boundary value problem have been obtained by this approach. The method of solution to flow in an aquifer by discrete kernel approach when a group of wells are operating has been explained here.

Let a confined aquifer which is homogeneous and isotropic be initially at rest condition. Let a unit impulse quantity is withdrawn at t = 0, from a fully penetrating well of very small diameter. The drawdown at a distance r at time t is known as unit impulse kernal and is given by

$$k\ (t) = \frac{e^{-r^2/4\beta t}}{4\pi T t} \ ; \quad \beta = T/\phi \qquad \ldots\ldots (1)$$

in which T = transmissivity and

ϕ = storage coefficient.

When a unit quantity of water (say 1 m^3) is withdrawn in the 1^{st} unit time period (one unit time period may be 1 hour, 4 hours, one day, one week etc.), expression for drawdown at the end of m^{th} unit time period can be derived making use of equation 1 and it is given by

$$\partial_r (m) = \int_0^1 \frac{e^{-r^2/4\beta (m-z)}}{4\pi T (m-z)} dz \quad ..... (2)$$

Where $\partial_r(m)$ is known as discrete kernal for drawdown. Discrete kernal is nothing but the response of a linear system when a unit pulse excitation is given to a system during the first unit time period.

Substituting $u = \frac{r^2}{4\beta (m-z)}$

$$dz = \frac{r^2}{4\beta u^2} du$$

in equation 2, $\partial_r(m)$ is found to be

$$\partial_r(m) = \frac{1}{4\pi T} [E_1(\frac{r^2}{4\beta m}) - E_1(\frac{r^2}{4\beta (m - 1)})] \quad ..... (3)$$

where $E_1(X)$ is an exponential integral defined as

$$E_1 (X) = \int_X^\infty \frac{e^{-Y}}{Y} dy.$$

For any positive integer values of m, $\partial_r(m)$ can be evaluated using equation 3. The transmissivity term appearing in equation 3 should have the dimension of $(\text{length})^2$ per unit time period. The exponential integral appearing in equation (3) can be evaluated in the following manner :

The exponential integral $E_1(X)$ for $0 \leq X \leq 1$ can be evaluated using the expression

$$E_1(X) + \log_e X = a_o + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4 + a_5 X^5 + \epsilon(X)$$

$$|\epsilon(X)| < 2 \times 10^{-7}$$

Where,

$a_o = -.57721\ 566$

$a_1 = .99999\ 193$

$a_2 = -.24991\ 055$

$a_3 = .05519\ 968$

$a_4 = .00976\ 004$

$a_5 = .00107\ 857$

for, $1 \leq X \leq \infty$

$$X e^X E_1(X) = \frac{X^2 + a_1 X + a_2}{X^2 + b_1 X + b_2}$$

Where,

$a_1 = 2.334733$

$a_2 = .250621$

$b_1 = 3.330657$

$b_2 = 1.681534$

Use of Discrete Kernel to Evaluate Drawdown for Variable pumping Rate.

Let water is whithdrawn which varies with time from a single well as shown in Fig.1. Let it is required to find drawdown at an observation point which is located at a distance r from the pumping well.

Making use of equation 1, the drawdown at the end of n^{th} unit time period can be written as

$$S_r(n) = \int_0^n \frac{Q(z) e^{-r^2/4\beta\,(n-z)}}{4\,\pi\,T\,(m-z)}\,dz \qquad \ldots\ldots (4)$$

Discretising the time span into n unit time step and assuming that within each time step, the withdrawal rate is constant but varies from step to step equation 4 can be written as

$$S_r(n) = Q(1) \int_0^1 \frac{e^{-r^2/4\beta\,(n-z)}}{4\,\pi\,T\,(n-z)}\,dz$$

$$+ Q(2) \int_1^2 \frac{e^{-r^2/4\beta\,(n-z)}}{4\,\pi\,T\,(n-z)}\,dz$$

$$+ Q(r) \int_{\gamma-1}^{\gamma} \frac{e^{-r^2/4\beta\,(n-z)}}{4\,\pi\,T\,(n-z)}\,dz$$

$$+ Q(n) \int_{n-1}^{n} \frac{e^{-r^2/4\beta\,(n-z)}}{4\,\pi\,T\,(n-z)}\,dz \qquad \ldots\ldots (5)$$

$$\text{or } S_r(n) = \sum_{\gamma=1}^{n} Q(\gamma) \int_{\gamma-1}^{\gamma} \frac{e^{-r^2/4\beta\,(n-z)}}{4\,\pi\,T\,(n-z)}\,dz \qquad \ldots\ldots (6)$$

Let $z - \gamma + 1 = Y$

or $z = Y + \gamma - 1$ and $dz = dy$

Making these substitutions in equation 6

$$S_r(n) = \sum_{\gamma=1}^{n} Q(\gamma) \int_0^1 \frac{e^{-r^2/4\beta(n-\gamma+1-Y)}}{4\pi T(n-\gamma+1-Y)} dY \quad .....(7)$$

or

$$S_r(n) = \sum_{\gamma=1}^{n} Q(\gamma)\, \partial(n-\gamma+1) \quad .....(8)$$

Thus drawdown at the end of first unit time,

$$S_r(1) = \partial(1).Q(1)$$

drawdown at the end of second unit time is

$$S_r(2) = Q(1).\,\partial(1) + Q(2).\partial(2).$$

Thus, $\partial(1)$, $\partial(2)$, ... $\partial(m)$ can be calculated and stored. Using these values the drawdown for any pumping pattern can be calculated. If several wells are operating, the corresponding discrete kernels can be calculated and using the superposition the resulting drawdowns can be calculated. Thus

$$S_r(n) = \sum_{i=1}^{P} \sum_{\gamma=1}^{n} Q_1(\gamma).\partial_i(n-\gamma+1)$$

Where P is the total number of pumps operating.

$$\partial_i(m) = \frac{1}{4\pi T}\left[E_1\left(\frac{r_i^2}{4\beta m}\right) - E_1\left(\frac{r_i^2}{4\beta(m-1)}\right)\right]$$

r_i = distance of i^{th} well from the observation point.

The discrete kernel approach is not limited to the rare situations when the pumping kernel function is known analytically. For heterogeneous aquifers, of finite size and intersected by a stream, the methodology has already been developed and implemented on the computer (morel-Seytoux and Daly, 1975).

The advantage of the methodology results from the following facts :

(a) A finite difference model is used only to generate basic response functions to specialized excitations in an aquifer without any stream interaction. Once these basic response functions have been calculated for a particular aquifer and saved, simulation of the aquifer behaviour to any pumping pattern is obtained without ever making use any longer of the numerical (e.g., finite difference) model.

(b) Because the finite difference model is used only to generate the response functions smaller grid sizes and time increments can be used to calculate accurately the influence coefficients than is usually feasible when performing a large mumber of simulation runs under many varied pumping patterns. Also with this procedure the accuracy of the calculations for an actual simulation remains that with which the influence coefficients were obtained. On the other hand in typical simulation approaches the accuracy of the finite-difference model is usually tested against an analytical solution using small time and space increments. When the simulator is used on an actual aquifer, vastly different time and space increments are used and the accuracy of the results is to a large degree unknown.

(c) Because the response functions are known explicitly in terms of the controllable (decision) variables many management problems can be solved through the efficient algorithms associated with a well structured Mathematical Programming formulation.

When the flow domain is nonhomogeneous and irregular and the boundary conditions and initial conditions are complex, numerical methods should be applied to solve the goundwater flow problem. The approach to arrive at the Bonssinesq's equation which describes unsteady groundwater flow in nonhomogeneous isotropic aquifer is described here.

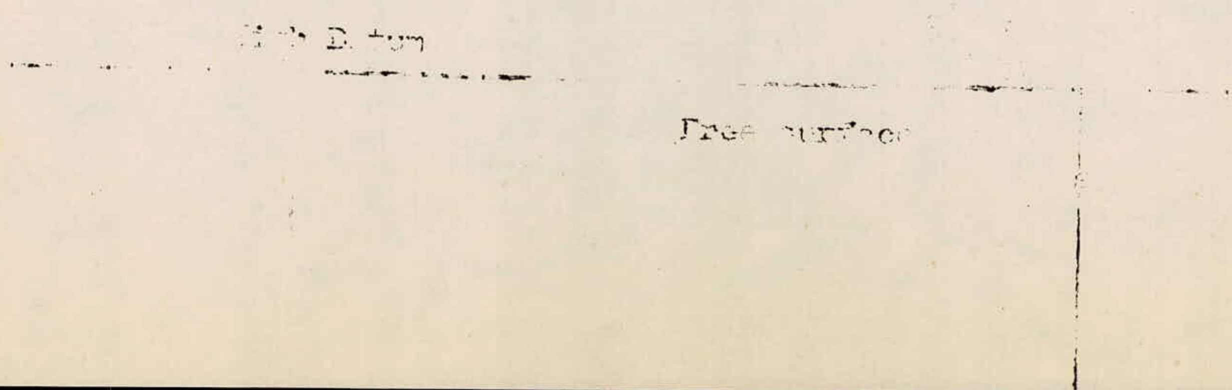
The basic equation which describes a two dimensional unsteady flow in an isotropic aquifer is derived as follows:

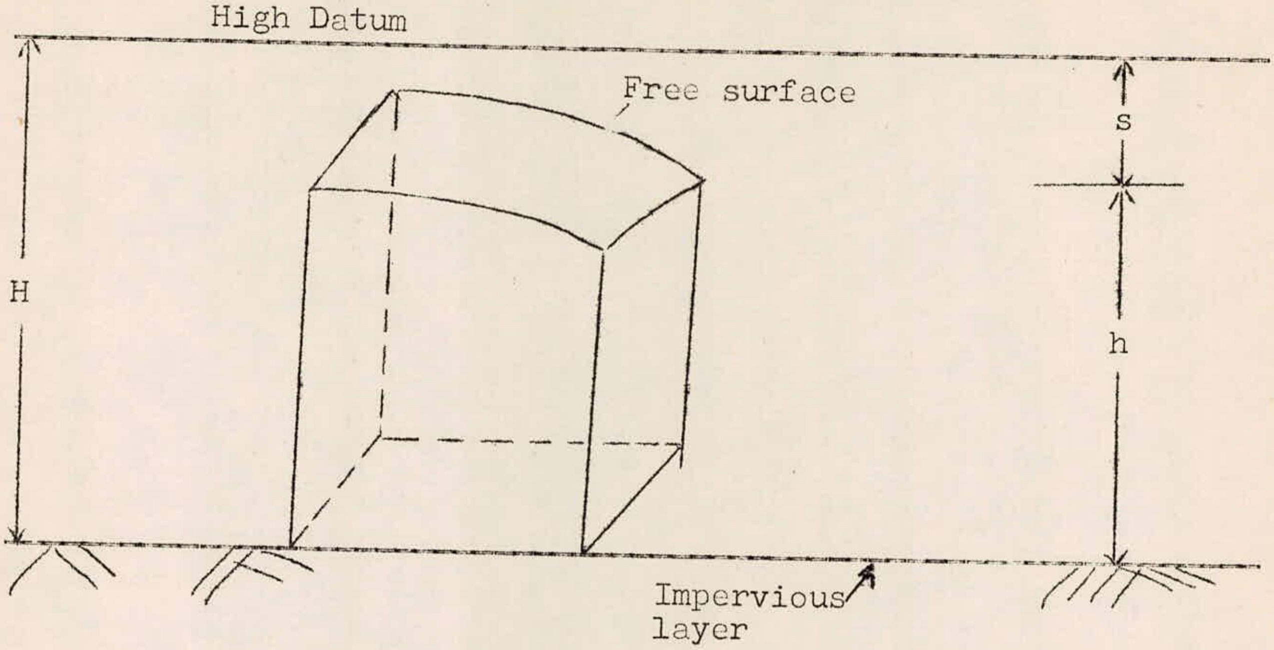
Consider a vertical column in the flow domain at some point x, y as shown in Figure 1. Let the flow conditions are such that the **Dupuit** assumptions stated below are valid.

Dupuit assumptions :

i) For small inclination of the line of seepage the stream lines can be taken as horizontal, and hence, the equipotential lines are vertical.

ii) The hydraulic gradient at any section is given by slope of the free surface and it remains invariant with depth.





The mass balance for an incompressible fluid for the period t to $t + \Delta t$ can be written as :

Total quantities of inflow in time Δt

= Total quantities of outflow in time Δt

+ quantities gone to storage. (1)

Total inflow in time span t

$$= q_x \cdot \Delta y \cdot \Delta t + q_y \cdot \Delta x \cdot \Delta t,$$

in which

q_x = quantity of flow in x direction per unit time per unit length in the direction of y

q_y = quantity of flow in y direction per unit time per unit length in the direction of x.

Total outflow in time span Δt

$$= \left(q_x + \frac{\partial q_x}{\partial x} \Delta x\right) \Delta y \cdot \Delta t$$

$$+ \left(q_y + \frac{\partial q_y}{\partial y} \Delta y\right) \Delta x \cdot \Delta t + q \, \Delta x \, \Delta y \, \Delta t$$

in which q is the net withdrawal from aqifer storage per unit area per unit time.

Let the average water table position changes from h to $h + \Delta h$ in time span Δt.

The quantities gone to storage is given by

$$\Delta h \cdot \phi \cdot \Delta x \cdot \Delta y$$

Where, ϕ is the specific yield for unconfinedaquifer and it is equal to storage coefficient for a confined aquifer.

Substituting the different terms in equation (1)

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + q + \phi \frac{\partial h}{\partial t} = 0 \qquad \ldots\ldots(2)$$

Since $H = h + s$, therefore, $\frac{\partial h}{\partial t} = - \frac{\partial s}{\partial t}$.

Replacing $\frac{\partial h}{\partial t}$ by $- \frac{\partial s}{\partial t}$, equation (2) reduces to

$$\phi \frac{\partial s}{\partial t} - \frac{\partial}{\partial x} (q_x) - \frac{\partial}{\partial y} (q_y) = q \qquad (3)$$

Let the flow domain is divided into a grid system as shown below:

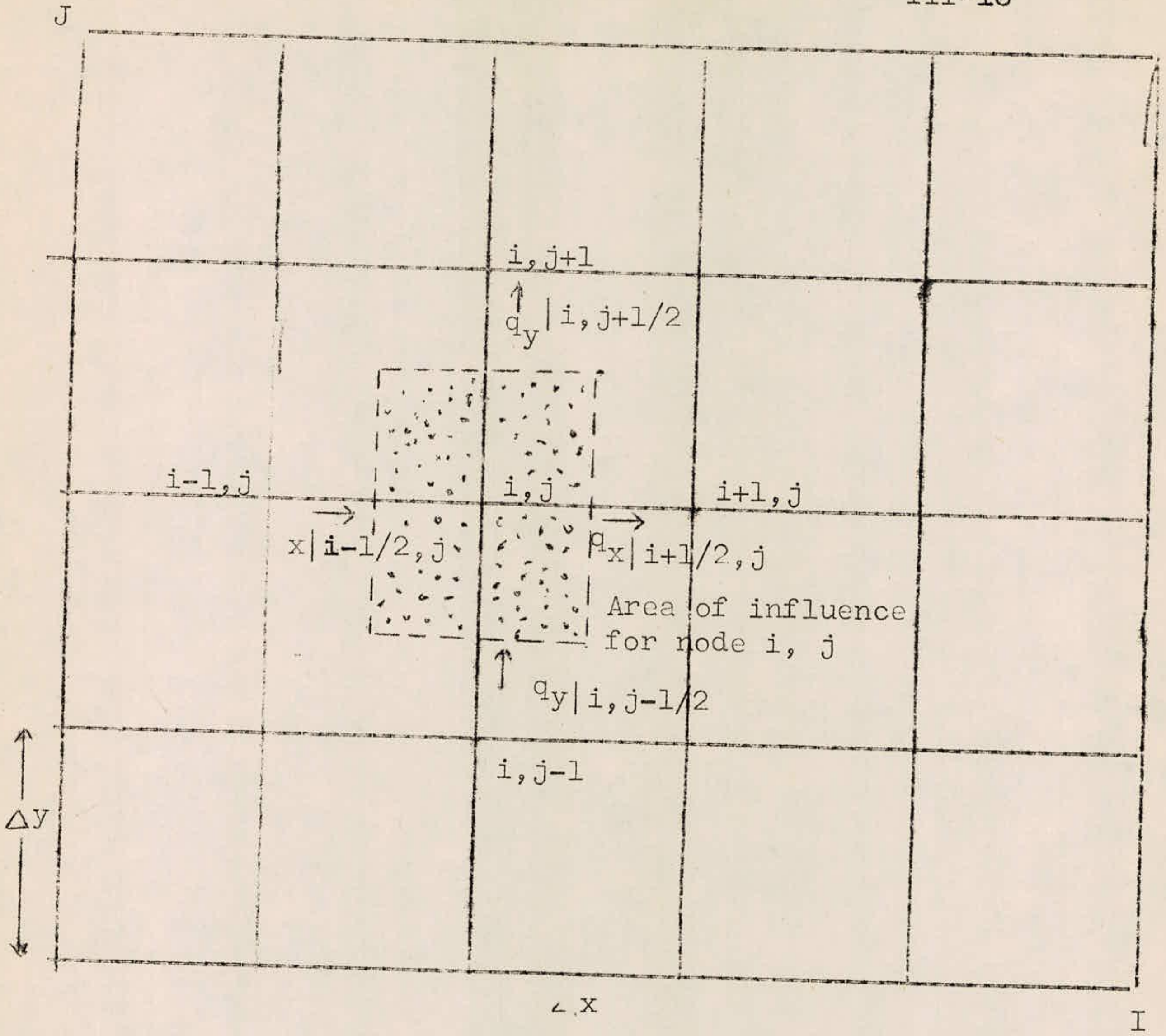


Figure 2

$\frac{\partial q_x}{\partial x}\Big|_{i,j}$ means $\frac{q_x\,|_{i+1/2,j} - q_x\,|_{i-1/2,j}}{\Delta x}$

$q_x\,|_{i+1/2,j}$ etc are given by :

$$q_x\,|_{i+1/2,j} = T_{i+1/2,j}\,\frac{[S_{i+1,j} - S_{i,j}]}{\Delta x}$$

$$T_{i+1/2,j} = \frac{2T_{i,j} \cdot T_{i+1,j}}{T_{i,j} + T_{i+1,j}}$$

$$q_x \big|_{i-1/2,j} = T_{i-1/2,j} \frac{[S_{i,j} - S_{i-1,j}]}{\Delta x}$$

$$T_{i-1/2,j} = \frac{2T_{i,j}\ T_{i-1,j}}{T_{i,j} + T_{i-1,j}}$$

$$\frac{\partial q_y}{\partial y}\bigg|_{i,j} \text{ means } \frac{q_y \big|_{i,j+1/2} - q_y \big|_{i,j-1/2}}{\Delta y}$$

$$q_y \big|_{i,j+1/2} = T_{i,j+1/2} \frac{(S_{i,j+1} - S_{i,j})}{\Delta y}$$

$$T_{i,j+1/2} = \frac{2T_i\ T_{j+1}}{T_i + T_{j+1}}$$

$$q_y \big|_{i,j-1/2} = T_{i,j-1/2} (S_{i,j} - S_{i,j-1})$$

$$T_{i,j-1/2} = \frac{2T_{i,j}\ T_{i,j-1}}{T_{i,j} + T_{i,j-1}}$$

The finite difference form of the equation at node i, j in an ADI scheme is as follows :

$$\phi_{i,j} \frac{[\overset{*}{s}_{i,j} - \overset{\circ}{s}_{i,j}]}{\Delta t} - \frac{1}{\Delta x^2} \Big\{ T_{i+1/2,j} [\overset{*}{s}_{i+1,j} - \overset{*}{s}_{i,j}]$$

$$- T_{i-1/2,j} [\overset{*}{s}_{i,j} - \overset{*}{s}_{i-1,j}] \Big\}$$

$$= \frac{1}{\Delta y^2} \Big\{ T_{i,j+1/2} [\overset{\circ}{s}_{i,j+1} - \overset{\circ}{s}_{i,j}]$$

$$- T_{i,j-1/2} [\overset{\circ}{s}_{i,j} - \overset{\circ}{s}_{i,j-1}] \Big\} + q_{i,j}$$

$$\frac{\phi_{i,j}[\overset{\gamma}{S}_{i,j} - \overset{*}{S}_{ij}]}{\Delta t} - \frac{1}{\Delta y^2}\left\{T_{i,j+1/2}\,[\overset{\gamma}{S}_{i,j+1} - \overset{\gamma}{S}_{i,j}] - T_{i,j-1/2}\,[\overset{\gamma}{S}_{i,j} - \overset{\gamma}{S}_{i,j-1}]\right\}$$

$$= \frac{1}{\Delta x^2}\left\{T_{i+1/2,j}\,[\overset{*}{S}_{i+1,j} - \overset{*}{S}_{i,j}] - T_{i-1/2,j}\,[\overset{*}{S}_{i,j} - \overset{*}{S}_{i-1}]\right\} + q_{i,j}$$

Thus one ADI cycle involves two successive time steps of duration Δt.

After writing the equation at all nodal points in a flow domain with known initial and boundary conditions the linear algebraic equations are solved to find the drawdown at different nodes at different time in succession starting from time step one. Such huge calculations are only possible by a computer.