Well Hydraulics

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INTRODUCTION

A sustainable and efficient utilization of the groundwater resources in a region needs to ensure

(i) high water supply accountability (specially during drought period);

(ii) high water quality (mandatory for drinking water);

(iii) low energy consumption (by controlling well spacing and pumping schedule) , and

(iv) low environmental impact (by restricting stream depletion due to excessive pumping).

Ground water can be tapped almost everywhere in a groundwater basin digging a well and water can be lifted to ground surface spending external energy except for a flowing well. An in-depth knowledge in well hydraulics is required for predicting response of an aquifer to pumping in a well under various hydro-geological conditions and estimating the safe yield of the well. The safe yield of a well is computed multiplying specific capacity at the stabilized drawdown with permissible drawdown in the well. In case of confined aquifer, the drawdown in piezometric surface should not cause the confined aquifer to become unconfined; and incase case of an unconfined aquifer, the water level in the well should be above the top of the well screen. A safety factor ranging from 60 to 80% is assumed while recommending the design discharge of the well. A basic knowledge in well hydraulics is also necessary in solving several well aquifer interaction related problems such as:

(i) Design of an optimal well field i.e. the number and spacing of the drainage wells and the pumping schedule to control water logging in an internally draining basin;

(ii) Estimation of artificial groundwater recharge through a vertical shaft filled with gravel and coarse material

(iii) Stream depletion due to pumping in a well in the vicinity of the stream;

(iv) Estimation of safe yield of a well;

(v) Estimation of transmissivity and storage coefficient from aquifer test data.

UNSTEADY FLOW TO A WELL (THEIS' SOLUTION)

For two dimensional axis symmetry radial flow in a confined aquifer the governing equation for unsteady groundwater flow is derived as follows: Consider a control volume bounded by cylindrical surfaces at $r-\Delta r/2$, and $r+\Delta r/2$, vertical surfaces at θ and at $\theta+\Delta\theta$ and horizontal impervious surfaces at the base and top of the confined aquifer (Fig.2). Let at point (r, θ), the radial velocity be v_r and hydraulic head be h.

$$\text{Inflow}=\left(r-\Delta r/2\right)\Delta\theta\, b\left[v_r-\frac{\partial v_r}{\partial r}(\Delta r/2)\right]\Delta t\ ;$$

Outflow= $\left(r + \Delta r/2\right)\Delta\theta\, b\left[v_r + \frac{\partial v_r}{\partial r}(\Delta r/2)\right]\Delta t$;

Change in storage= $S\; r\; \Delta\theta\; \Delta r\; b\; \Delta h$.

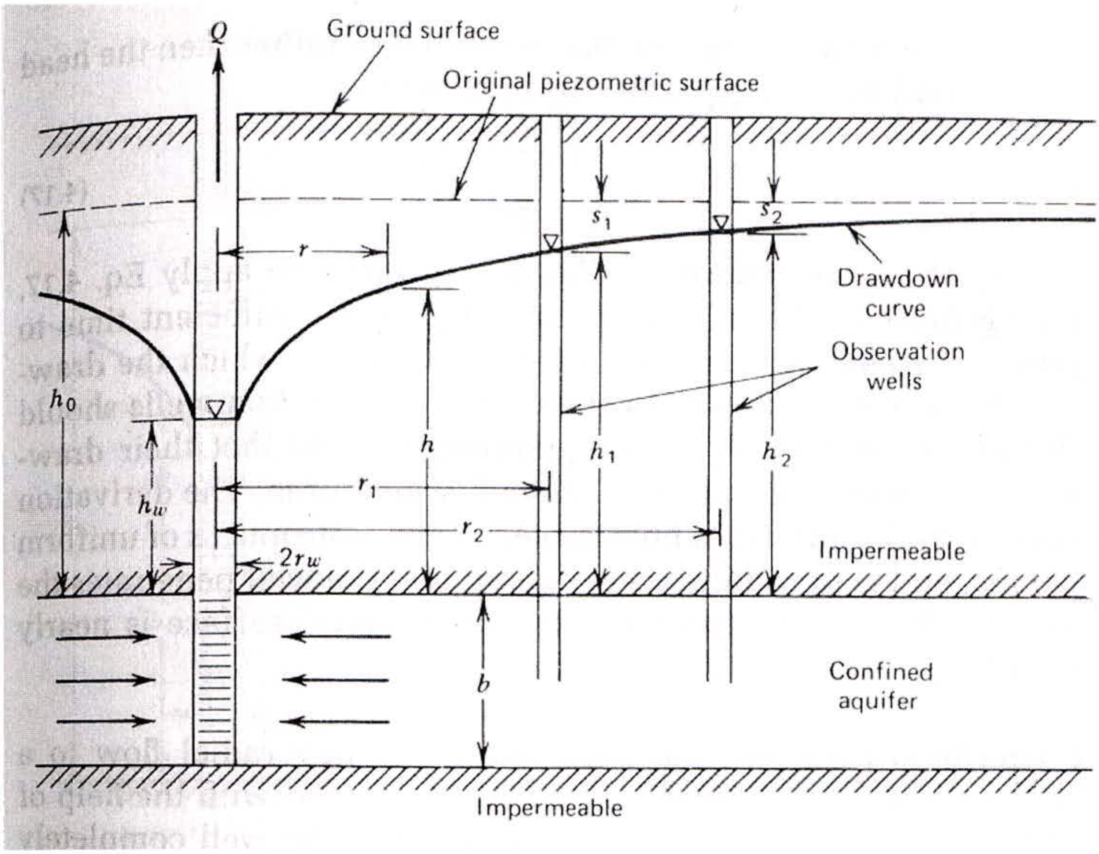

Fig. 1 Unsteady flow to a well in an extensive confined aquifer (Todd, 1980)

Performing mass balance in the control volume and incorporating Darcy's law, $v_r = -k\frac{\partial h}{\partial r}$, we derive the governing differential equation for axis-symmetry radial flow in a confined aquifer

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{\phi}{T}\frac{\partial h}{\partial t} \tag{1}$$

where $T = kb$ = transmissivity; $\phi = Sb$ =storage coefficient.

The position of water table or piezometric surface is generally measured from a high datum instead of measuring its position from the base of the aquifer. For an aquifer initially at rest, the initial piezometric surface is selected as a high datum. At any time, at a location, the sum of draw down in piezometric surface measured from high datum and height of the surface measured from low datum is constant i.e s+h = a constant. The above equation can be expressed as:

$$\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}=\frac{\phi}{T}\frac{\partial s}{\partial t} \tag{2}$$

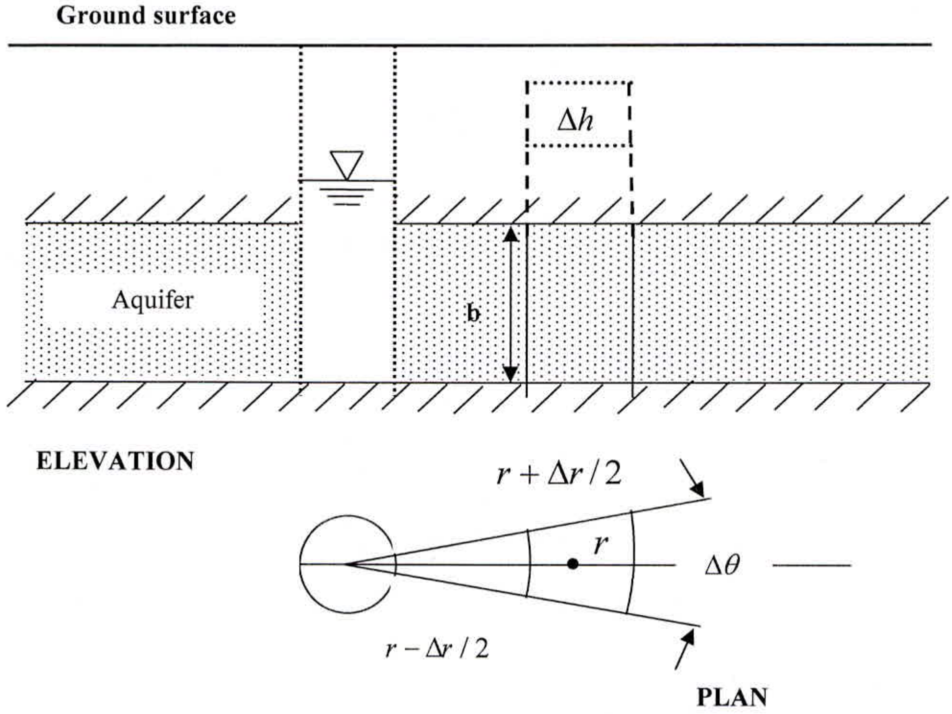

Fig. 2 Mass balance in a control volume

When a fully penetrating well in an extensive homogeneous and isotropic confined aquifer is pumped, axis symmetry radial flow condition prevails and the unsteady flow is governed by above equation. Drawdown consequent to pumping water from a well of very small radius at a constant rate is predicted using Theis' solution for the above differential equation satisfying the initial condition $s(r,0)=0$, and the boundary conditions (i) $s(\infty,t)=0$; (ii) $2\pi\, kb\left(r\frac{\partial s}{\partial r}\right)_{r\to 0}=-Q_P$. The first boundary condition is a constant head boundary condition and is known as Dirichelt type and the second boundary condition is known as Cauchy type boundary condition. Theis' non-equilbrium solution is:

$$s(r,t)=\frac{Q_P}{4\pi T}\int_{\frac{r^2\phi}{4Tt}}^{\infty}\frac{e^{-v}}{v}dv=\frac{Q_P}{4\pi T}W(X) \tag{3}$$

$X=\frac{r^2\phi}{4Tt}$, r = distance of the piezometer (observation well) from the well, t=time since pumping starts. $W(X)$ is known as Theis' Well Function with argument X

$$W(X)=\int_X^{\infty}\frac{e^{-v}}{v}dv=-0.57721-\ln(X)-\sum_{n=1}^{\infty}\frac{(-1)^n X^n}{nn!} \tag{4}$$

The convergence of the series is very slow for $X \geq 1$. The well function $W(X)$ can be computed using the following polynomial approximation.

For $X \leq 1$

$$W(X) = -\ln(X) - 0.57721566 + 0.99999193X - 0.24991055X^2 + 0.05519968 * X^3 - 0.00976004X^4 + 0.00107857X^5 \quad (5)$$

For $X > 1$

$$Xe^X W(X) = \frac{X^4 + 8.5733287X^3 + 18.059017X^2 + 8.6347608X + 0.26777373}{X^4 + 9.5733223X^3 + 25.632956X^2 + 21.099653X + 3.9584969} \quad (6)$$

If pumping stops after time t_p, the residual drawdown at time t is given by:

$$s(r,t) = \frac{Q_P}{4\pi T}\left[W\left(\frac{\varphi r^2}{4Tt}\right) - W\left(\frac{\varphi r^2}{4T(t-t_p)}\right)\right] \quad (7)$$

Let unit volume of water be pumped in a time period Δt, and the pumping be discontinued, at time $t = n\Delta t$, where n is an integer. In that case the pumping rate, $Q_P = \frac{1}{\Delta t}$. Let the drawdown $s(r, n\Delta t)$ corresponding to this unit pulse withdrawal be designated as $\delta(r, n\Delta t)$, known as the discrete kernel coefficient.

$$\delta(r, n\Delta t) = \frac{1}{4\pi T\Delta t}\left[W\left(\frac{\phi r^2}{4Tn\Delta t}\right) - W\left(\frac{\phi r^2}{4T(n\Delta t - \Delta t)}\right)\right] \quad (8)$$

If pumping rate is not uniform or pumping is not continuous, the drawdown is computed using the relation

$$s(r, n\Delta t) = \sum_{\gamma=1}^{n} Q_V(\gamma)\delta\{r, (n-\lambda+1)\Delta t\} \quad (9)$$

Where $Q_V(\gamma)$=volume of water withdrawn during γ^{th} Δt time unit.

Specific Capacity

Specific capacity is an indicator of well condition. The specific capacity of the well is estimated imposing a steady state flow condition. It is defined as pumping rate per unit drawdown:

$$S_C = \frac{Q_p}{S_w} \quad (10)$$

where, $S_C(m^2/day)$ = specific capacity; $Q_p(m^3/day)$ = uniform pumping rate, $S_w(m)$ = stabilized drawdown in the well corresponding to the pumping rate. We derive specific capacity for a fully penetrating well, the well screen intercepting the entire thickness of a homogeneous, isotropic confined and finite aquifer. Under continuous constant pumping, the flow to a well in a finite aquifer having a constant head boundary at the outer periphery attains a steady state condition. The aquifer is assumed to be a circular island and a single well is located at the centre of the circular island. The stabilized drawdown is computed considering the flow to be non-linear.

Darcy's Law

Darcy's law states that in a homogeneous isotropic porous medium, the discharge per unit of pore plus soil area, which is known as discharge velocity , v, is linearly proportional to the hydraulic gradient, i, i.e., $v = -k\dfrac{dh}{ds} = ki$, where $\dfrac{dh}{ds}$ = variation in hydraulic head along the flow direction, s; k = coefficient of permeability. Several investigators have found that, for $1 < R_e < 12$, flow in soils changes from laminar to turbulent (vide Harr, 1962), where the Reynolds' number in case of flow through soils is given by $R_e = vd\rho/\mu$, $v =$ discharge velocity, $d =$ average of diameters of soil particles, $\rho =$ density of fluid, and μ = coefficient of viscosity of water.

Forchheimer's Law

For a fully turbulent flow condition, Forchheimer has proposed the following relation between the hydraulic gradient and discharge velocity:

$$i = av + bv^2 \tag{11}$$

where 'a' , and 'b' are positive constants and the hydraulic gradient $i = -dh/ds$. Equation (11) is valid for which dh/ds is negative or the hydraulic gradient i as well as the velocity v is positive. Such situation prevails when recharge is taking place through a well. Solving the quadratic equation, the velocity in terms of hydraulic gradient in turbulent flow condition is given by:

$$v = \frac{-a}{2b} + \frac{\sqrt{a^2 + 4bi}}{2b} = \frac{-a}{2b} + \frac{a}{2b}\left[1 + \frac{4bi}{a^2}\right]^{1/2} \cong \frac{i}{a} - \frac{bi^2}{a^3} \tag{12}$$

The other root, which yields a negative velocity, is dropped as the velocity is positive. Equation (12) indicates that for a given gradient i, under turbulent flow condition the velocity is less than that under laminar flow condition.

When an aquifer is pumped, the hydraulic gradient, i, as well as the radial velocity is negative, the flow direction being opposite to the radial direction. Forchheimer's equation in such situation is

$$-i = -av + bv^2 \tag{13}$$

Solving the quadric equation

$$v = \frac{a}{2b} - \frac{\sqrt{a^2 - 4bi}}{2b} = \frac{a}{2b} - \frac{a}{2b}\left[1 - \frac{4bi}{a^2}\right]^{1/2} = \frac{i}{a} + \frac{bi^2}{a^3} \tag{14}$$

The other root is dropped as the velocity is negative

Based on experiments Abbood (2009) has recommended the following empirical relation for predicting a and b:

$$\frac{a}{a^*} = 2.64\frac{n^{1/4}}{d^2}\times 1.028^{25-T} \quad ; \quad \frac{b}{b^*} = 2.514\frac{n^{1.333}}{d} \tag{15a,b}$$

where T = temperature (°C),

$a^* = 31.2(s/m)$

$b^* = 1513\left(s^2/m^2\right)$

Specific Recharge Capacity

From equation (11),

$$bv_r^2 + av_r + \frac{dh}{dr} = 0 \tag{16}$$

Under steady state flow condition, at any radial distance 'r' from the well, the radial flow is given by:

$$Q_r = 2\pi\, rD\, v_r = Q_R \tag{17}$$

where, D = thickness of aquifer, v_r = radial velocity.

Incorporating $v_r = \dfrac{Q_R}{2\pi\, rD}$ in equation (16)

$$b\left(\frac{Q_R}{2\pi\, D}\right)^2\frac{1}{r^2} + \left(\frac{a\ Q_R}{2\pi\, D}\right)\frac{1}{r} = -\frac{dh}{dr} \tag{18}$$

Integrating and applying the boundary condition, $h(r_w) = h_w$, the rise at any radial distance r is given by

$$h(r) = h_w - a\frac{Q_R}{2\pi D}\log_e\left(\frac{r}{r_w}\right) - b\left(\frac{Q_R}{2\pi D}\right)^2\left[\frac{1}{r_w} - \frac{1}{r}\right] \tag{19a}$$

The first part of head loss, $a\dfrac{Q_R}{2\pi D}\log_e\left(\dfrac{r}{r_w}\right)$, is linear aquifer loss and the second part, $b\left(\dfrac{Q_R}{2\pi D}\right)^2\left[\dfrac{1}{r_w} - \dfrac{1}{r}\right]$, is non linear aquifer loss component.

A true flow domain is generally extensive. In case of a confined extensive aquifer, theoretically steady state condition is not reached whether water is pumped into or pumped out. Let us consider that at an unknown radial distance R_i the hydraulic gradient is very small and equal to 0.001.

Incorporating $r = R_i$ and $\dfrac{dh}{dr} = -0.001$ in equation (18) and simplifying

$$0.001R_i^2 - \left(\frac{a\ Q_R}{2\pi\ D}\right)R_i - b\left(\frac{Q_R}{2\pi\ D}\right)^2 = 0 \tag{20}$$

Solving the quadratic equation, the radius of influence corresponding to the recharge rate is given by:

$$R_i = \frac{\left(\frac{a\ Q_R}{2\pi\ D}\right) + \sqrt{\left(\frac{a\ Q_R}{2\pi\ D}\right)^2 + 0.004b\left(\frac{Q_R}{2\pi\ D}\right)^2}}{0.002} \tag{21}$$

As R_i is positive, the other root is not applicable. Corresponding to $r = R_i$, equation (19a) reduces to

$$h_w - h(R_i) = a\frac{Q_R}{2\pi D}\log_e\left(\frac{R_i}{r_w}\right) + b\left(\frac{Q_R}{2\pi D}\right)^2\left[\frac{1}{r_w} - \frac{1}{R_i}\right] \tag{22}$$

The radial distance R_i will be very large compared to well radius. Neglecting the term $\frac{1}{R_i}$

$$h_w - h(R_i) = a\frac{Q_R}{2\pi D}\log_e\left(\frac{R_i}{r_w}\right) + \frac{b}{r_w}\left(\frac{Q_R}{2\pi D}\right)^2 \tag{23}$$

For known injection rate Q_R and R_i computed from equation (21), the corresponding seepage head $h_w - h(R_i)$ is predicted from equation (23). The specific recharge is given by $\frac{Q_R}{h_w - h(R_i)}$.

In case of a pumping well, $\frac{dh}{dr}$ is positive, therefore, the hydraulic gradient, *i*, which is equal to $-\frac{dh}{dr}$, is negative. The radial velocity v_r is also negative. Accordingly Forchheimer equation is written as

$$-i = -av_r + bv_r^{\ 2} \tag{24}$$

$$2\pi rDv_r = -Q_P \tag{25}$$

$$b\left(\frac{Q_P}{2\pi\ rD}\right)^2 + a\frac{Q_P}{2\pi rD} = \frac{dh}{dr} \tag{26}$$

Integrating

$$h = -\left(\frac{Q_P}{2\pi D}\right)^2\frac{b}{r} + \frac{aQ_P}{2\pi D}\ln r + A \tag{27}$$

Let h be equal to h_w at $r = r_w$. Accordingly the constant A is found to be

$$A = h_w + \left(\frac{Q_p}{2\pi D}\right)^2 \frac{b}{r_w} - \frac{aQ_p}{2\pi D}\ln r_w \tag{28}$$

The head $h(r)$ at radial distance r

$$h(r) = \frac{aQ_p}{2\pi D}\ln\frac{r}{r_w} + h_w + \left(\frac{Q_p}{2\pi D}\right)^2\left(\frac{b}{r_w} - \frac{b}{r}\right) \tag{29}$$

$$h(R_i) - h_w = \frac{a\,Q_p}{2\pi D}\ln\frac{R_i}{r_w} + b\left(\frac{Q_p}{2\pi D}\right)^2\left(\frac{1}{r_w} - \frac{1}{R_i}\right) \tag{30}$$

The first part of the head loss is aquifer loss attributed to laminar flow and the second part is the aquifer loss attributed to non-linear turbulent flow. The specific capacity of the pumping well is thus reduced due to turbulence.

Applying the boundary condition at $r = R_i, \frac{dh}{dr} = 0.001$, in equation (26) we obtain

$$0.001R_i^2 - a\frac{Q_p}{2\pi D}R_i - b\left(\frac{Q_p}{2\pi\ D}\right)^2 = 0 \tag{31}$$

which is same as equation (20). The specific pumping capacity and recharge capacity are same.

REFERENCES

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