International Conference "Water, Environment, Energy and Society" (WEES–2009)
New Delhi, 12–16 January 2009

Modeling Rainfall-Runoff Process Using Grey System Theory with Two Solution Procedures

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ABSTRACT: Rainfall-runoff is a nonlinear process and can be modeled with a nonlinear modeling framework. The process of rainfall to runoff transformation involves a complex mechanism and it is quite difficult to understand too. To model the process accurately a large number of data set is required. While in the case of developing countries *viz.*, India data scarcity is major hindrance among the researchers and scientists. To overcome the difficulty of scanty data set and to get a reasonably good estimate of output (runoff) with available data, it is needed to work with some data driven approaches. These approaches consider only input and output, and avoid the inherent complex mechanism. In this context Grey system theory provides an approach to model a system considering only rainfall as input and gives runoff as the output from the system. Keeping above in view grey system theory approach is proposed in the present study. Two different solution procedures have been applied and results were compared.

INTRODUCTION

This paper describes the two different solution procedures for modeling rainfall-runoff process using grey system theory. Methods used to estimate runoff from rainfall are frequently classified into two groups-black box models and process models. In the black box modeling approach, empirical equations are used to relate runoff and rainfall, and only the input and output have physical meanings. Simple mathematical equations and time series methods fall into this category. Process models attempt to simulate the hydrological processes in a catchment and involve the use of many partial differential equations of continuity for surface and subsurface flow (Chiew *et al.*, 1993). Another modeling approach based on grey system theory is relatively new approach to model rainfall-runoff process. A system having partial information known is called as grey system. Without restoring to forming a knowledge base, the grey modeling scheme constructs a differential equation to characteristic the controlled system and therefore, next output from the model can be obtained by simple solving the differential equation. Grey modeling is based on Accumulated Generation Operation (AGO) series rather than using raw data, which results reduction in noise (Deng, 1989). Transformation of raw data series to Accumulated Generating Operation (AGO) series of different orders were shown in Table 1.

Many researchers (Xia, 1989; Lee and Wang, 1998; Yu *et al.*, 2000; Yu *et al.*, 2001; Trivedi and Singh, 2005a; Trivedi and Singh, 2005b) have been successfully applied grey system theory to model rainfall-runoff process. This paper deals with two different solution procedures; one uses the Lapalace transformation and convolution integral and other uses grey mathematics.

STUDY AREA

The present study has been carried out in Kothuwatari watershed of Tilaiya dam catchment in upper Damodar Valley, Jharkhand, India (Figure 1). The Kothuwatari watershed is located in between 24°12′27″N and 24°16′54″N latitudes and 85°24′18″E and 85°28′10″E longitudes. The watershed with an area of 27.93 km^2 is irregular in shape. The Kothuwatari watershed was selected for Watershed Management Programme of Indo-German Bilateral Project (IGBP) in the year 1991 for assessing the effect of various soil conservation measures.

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Table 1: Transformation of Raw Data Series to Accumulated Generating Operation (AGO) Series of Different Orders (Trivedi and Singh, 2005a)

Raw Data Series, r^0	*First Order AGO (1-AGO) Series, r^1*	*Second Order AGO (2-AGO) Series, r^2*	–	*p^{th} Order AGO (p-AGO) Series, r^p*
r_1^0	$r_1^1 = r_1^0$	$r_1^2 = r_1^1$	–	$r_1^P = r_1^{P-1}$
r_2^0	$r_2^1 = r_1^0 + r_2^0$	$r_2^2 = r_1^1 + r_2^1$	–	$r_2^P = r_1^{P-1} + r_2^{P-1}$
r_3^0	$r_3^1 = r_1^0 + r_2^0 + r_3^0$	$r_3^2 = r_1^1 + r_2^1 + r_3^1$	–	$r_3^P = r_1^{P-1} + r_2^{P-1} + r_3^{P-1}$
. . .	. . .	. . .		. . .
r_N^0	$r_N^1 = r_1^0 + r_2^0 + \ldots + r_N^0$	$r_N^2 = r_1^1 + r_2^1 + \ldots + r_N^1$	–	$r_N^P = r_1^{P-1} + r_2^{P-1} + \ldots + r_N^{P-1}$

Recorded rainfall and runoff data at Karso gauging station under this project by Damodar Valley Corporation, Hazaribagh, India have been used for this study. Sixteen storm events were considered in the present study. Out of these, ten storm events were included for model calibration and remaining six storm events were considered for model verification and validation purposes.

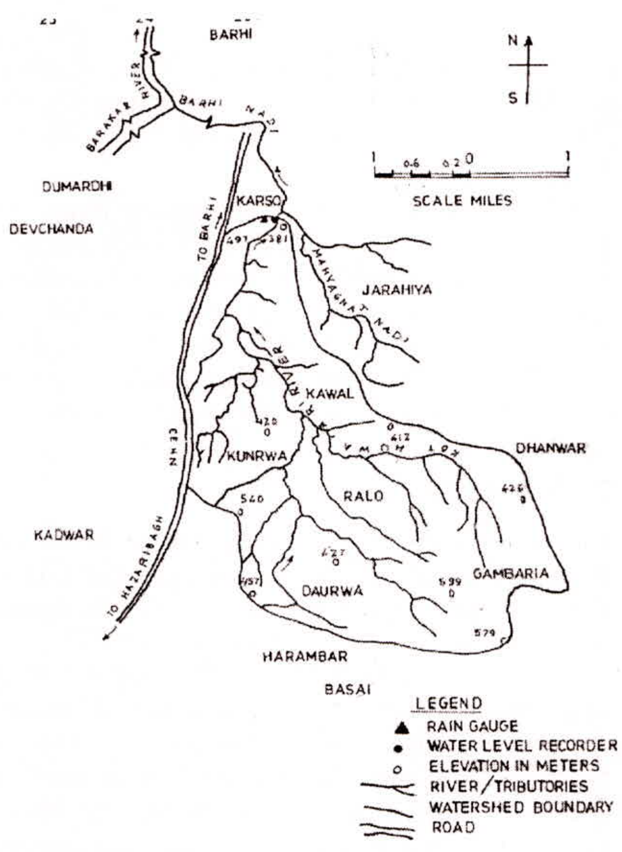

Fig. 1: Map of Kothuwatari Watershed, Jharkhand, India

STRUCTURE OF THE GREY MODEL

The causal relationship between rainfall and runoff over a catchment area, governed by the grey differential equation, can be expressed as follows (Xia, 1989),

$$\frac{d}{dt}Q^1(t) + \otimes_{a1} Q^1(t) = \otimes_{b0} \frac{d}{dt} R^1(t) + \otimes_{b1} R^1(t) \quad \ldots (1)$$

where $Q^1(t)$ and $R^1(t)$ are the first order time series of runoff and rainfall respectively, that is, $Q^1(t) = (q^1(1), q^1(2), q^1(3) \ldots q^1(t))$ and $R^1(t) = (r^1(1), r^1(2), r^1(3) \ldots r^1(t))$ in which, q^1 and r^1 are the first order accumulated generating operation (1^{th} – AGO) series of runoff and rainfall, and defined respectively as,

$$q^1(t) = \sum_{i=1}^{t} q^0(i) \quad \ldots (2)$$

$$r^1(t) = \sum_{i=1}^{t} r^0(i) \quad \ldots (3)$$

where q^0 and r^0 are the raw data of runoff and rainfall. $\otimes_{a1}, \otimes_{b0}, \otimes_{b1}$ are grey parameters.

Solution Approach I

Applying Laplace transforms to the Eqn. (1),

$$L\left[\frac{d}{dt}Q^1(t)\right] + \otimes_{a1}.L\left[Q^1(t)\right] = \otimes_{b0}. L\left[\frac{d}{dt}R^1(t)\right] + \otimes_{b1}.L\left[R^1(t)\right] \quad \ldots (4)$$

But Laplace transforms of derivatives are defined as,

$$L\left[\frac{d}{dt}Q^1(t)\right] = s.L\left[Q^1(t)\right] - Q^1(0) \quad \ldots (5)$$

where, s is a Laplace variable. Substituting the Laplace derivatives in Eqn. (4),

$$s.L\left[Q^1(t)\right]-Q^1(0)+\otimes_{a1}.L\left[Q^1(t)\right]=\otimes_{b0}.$$
$$[s.L\,(R^1(t))-R^1(0)]+\otimes_{b1}.\,L[R^1(t)] \quad \dots (6)$$

On arranging the terms of Eqn. (6) and simplifying, we get,

$$L\left[Q^1(t)\right]=\left[Q^1(0)-\otimes_{b0}.R^1(0)\right].L\left[e^{-\otimes_{a1}t}\right]+\otimes_{b0}.L\left[R^1(t)\right]+\left[\otimes_{b1}-\otimes_{b0}.\otimes_{a1}\right]\times L\left[e^{-\otimes_{a1}t}\right].L\left[R^1(t)\right] \quad \dots (7)$$

Applying inverse Laplace tranformation and convolution integral to Eqn. (7) yields,

$$Q^1(t)=\left[Q^1(0)-\otimes_{b0}R^1(0)\right]e^{-\otimes_{a1}t}+\otimes_{b0}R^1(t)+\left[\otimes_{b1}-\otimes_{b0}\otimes_{a1}\right]\int_0^t e^{-\otimes_{a1}\tau}R^1(t-\tau)d\tau \quad \dots (8)$$

Solution Approach II

Equation (1) can also be solved by using grey mathematics. For discrete data, the whitening of grey derivatives of Eqn. (1) can be expressed as,

$$\frac{d}{dt}Q^1(t)=\alpha^1\left[q^1(t)\right]=q^0(t) \quad \dots (9)$$

where α^1 is the first order inverse accumulated generating operator (1-IAGO) and t represents the time index,

$$\text{Similarly, } \frac{d}{dt}R^1(t)=r^0(t) \quad \dots (10)$$

For discrete data, $Q^1(t)$ and $R^1(t)$ may be defined as (Lee & Wang, 1998),

$$Q^1(t)=\frac{1}{2}\left[q^1(t)+q^1(t-1)\right] \quad \dots (11)$$

$$R^1(t)=r^1(t) \quad \dots (12)$$

Substituting Eqns. (9)–(12) in Eqn. (1) results in the following equation,

$$q^0(t)+\frac{\otimes_{a1}}{2}\left[q^1(t)+q^1(t-1)\right]=\otimes_{b0}\left[r^0(t)\right]+\otimes_{b1}r^1(t) \quad \dots (13)$$

By inverse accumulated generating operation (IAGO),

$$q^0(t)=q^1(t)-q^1(t-1) \quad \dots (14)$$

$$r^0(t)=r^1(t)-r^1(t-1) \quad \dots (15)$$

Substitution of Eqs. (14) and (15) in Eqn. (13) yields,

$$q^1(1)-q^1(t-1)+\frac{\otimes_{a1}}{2}\left[q^1(t)+q^1(t-1)\right]=\otimes_{b0}\left[r^1(t)-r^1(t-1)\right]+\otimes_{b1}r^1(t) \quad \dots (16)$$

Now, rearranging the Eqn. (16), we get,

$$q^1(t)=\frac{2-\otimes_{a1}}{2+\otimes_{a1}}q^1(t-1)+\frac{2(\otimes_{b0}+\otimes_{b1})}{2+\otimes_{a1}}r^1(t)-\frac{2\otimes_{b0}}{2+\otimes_{a1}}r^1(t-1) \quad \dots (17)$$

MODEL PARAMETERS ESTIMATION

According to grey system theory for discrete data, the whitening of grey derivatives of Eqn. (1) can be expressed as,

$$\frac{d}{dt}Q^1(t)\bigg|_{t=k}=\left[q^0(k)\right] \quad \dots (18)$$

For discrete data, $Q^1(t)$ and $R^1(t)$ may be defined as (Lee & Wang, 1998),

$$Q^1(t)\Big|_{t=k}=\frac{1}{2}\left[q^1(k)+q^1(k-1)\right] \quad \dots (19)$$

$$R^1(t)\Big|_{t=k}=r^1(k) \quad \dots (20)$$

Substituting of grey derivatives and Eqns. (19) and (20) in Eqn. (1) yields,

$$q^0(k)+\frac{\otimes_{a1}}{2}\left[q^1(k)+q^1(k-1)\right]=\otimes_{b0}\,r^0(k)+\otimes_{b1}\,r^1(k) \quad \dots (21)$$

Equation (21) may be expressed in a matrix form for k = 2, 3, 4, ..., n. The least square method has been used for estimation of model parameters as,

$$\begin{bmatrix}\hat{\otimes}_{a1}\\ \hat{\otimes}_{b0}\\ \hat{\otimes}_{b1}\end{bmatrix}=\left[\boldsymbol{U}^T\boldsymbol{U}\right]^{-1}\boldsymbol{U}^T\boldsymbol{Q} \quad \dots (22)$$

where, Q is the runoff matrix, U is the rainfall-runoff matrix, $\hat{\otimes}_{a1}$, $\hat{\otimes}_{b0}$ and $\hat{\otimes}_{b1}$ are estimators or identified values of $\otimes_{a1}$, $\otimes_{b0}$ and $\otimes_{b1}$ respectively, and T is the transpose operator.

CRITERIA FOR JUDGING THE DEVELOPED MODELS

Four criteria to judge the models performance, were employed in this study, as given next.

1. Revised Theil inequality coefficient (RTIC),

$$RTIC = \sqrt{\frac{\sum_{t=1}^{N}(O_t - P_t)^2}{\sum_{t=1}^{N} O_t}} \quad \text{... (23)}$$

2. Integral square error (ISE),

$$ISE = \frac{\sqrt{\sum_{t=1}^{N}(O_t - P_t)^2}}{\sum_{t=1}^{N} O_t} \times 100 \quad \text{... (24)}$$

3. Correlation coefficient (CC),

$$CC = \frac{\left[N\sum_{t=1}^{N}(O_t \times P_t) - \left(\sum_{t=1}^{N} O_t\right)\left(\sum_{t=1}^{N} P_t\right)\right]}{\sqrt{\left[N\sum_{t=1}^{N}(O_t)^2 - \left(\sum_{t=1}^{N} O_t\right)^2\right]\left[N\sum_{t=1}^{N}(P_t)^2 - \left(\sum_{t=1}^{N} P_t\right)^2\right]}} \quad \text{... (25)}$$

4. Coefficient of efficiency (CE),

$$CE = \left[1 - \frac{\sum_{t=1}^{N}(O_t - P_t)^2}{\sum_{t=1}^{N}(O_t - \bar{O})^2}\right] \times 100 \quad \text{... (26)}$$

where N is the number of observations, O_t is the observed value at time t and P_t is the predicted value at time t.

RESULTS AND DISCUSSION

The differential hydrological grey models as shown in Eqs. (8) and (17) were developed and calibrated for study area by using grey system theory to predict runoff on storm basis. The least square method was applied for the estimation of the model parameters. The average values of the model parameters *i.e.* $\otimes_{a1}$, $\otimes_{b0}$ and $\otimes_{b1}$ for the study area were found to be 0.299868, –0.536347, and 2.061873, respectively. On substituting the average value of the model parameters in Eqs. (8) and (17), the developed models for study area for predicting the storm runoff were obtain and were further verified for its performance. From the model response, it is concluded that all the three parameters vary with factors such as rainfall, runoff and watershed characteristics for each storm event. The parameter $\otimes_{a1}$ signifies the storage capacity and the physiographical factors of the grey process. A positive value of it explains the grey response function in decay of the above watershed characteristics. Further, $\otimes_{b0}$, which is the most sensitive parameter, depends on the temporal and spatial variation of the rainfall. The negative value of the parameter $\otimes_{b0}$ signifies the convex nature of the rainfall hyetograph. Similarly, $\otimes_{b1}$, which is more stable parameter for flood analysis, is pertinent to the overall meteorological condition of the study area during the flood event.

The goodness of fit of a predicted runoff hydrograph to an observed one was tentatively compared by qualitative evaluation. Visual comparison of observed and predicted storm runoff hydrographs were done for qualitative evaluation. Observed and predicted storm runoff hydrographs for the storm events for both the developed models used for verification have been shown in Figure 2. A close agreement between rising and recession segments of the observed and predicted hydrographs can be seen from figures. The quantitative performance is ascertained by a number of statistical indices such as Revised Theil Inequality Coefficient (RTIC), Integral Square Error (ISE), Correlation Coefficient (CC), coefficient of efficiency (CE). The quantitative performance indices for developed models were presented in Table 2. It can be seen from the table that the developed model-I yielded low value of the error indices in comparison to model-II, considered in the study for all the storm events. The correlation values more than 0.96 (for model-I) and 0.87 (for model-II) and values of coefficient of efficiency higher than 84% (for model-I) and 73% (for model-II) show of the suitability of both the developed models for runoff prediction.

CONCLUSIONS

Grey system theory approach is applied to model rainfall-runoff process in the present study. Two different solution procedures have been applied. Under quantitative evaluation, the lower values of various error indices and higher values of correlation indices confirm the models ability to predict storm runoff with reasonable accuracy for the study area. Some deviations between observed and predicted ordinates of storm runoff, as evident from the values of different performance indices, may be due to errors in collection of hydrological data and/or assumptions involved in formulation and development of the model.

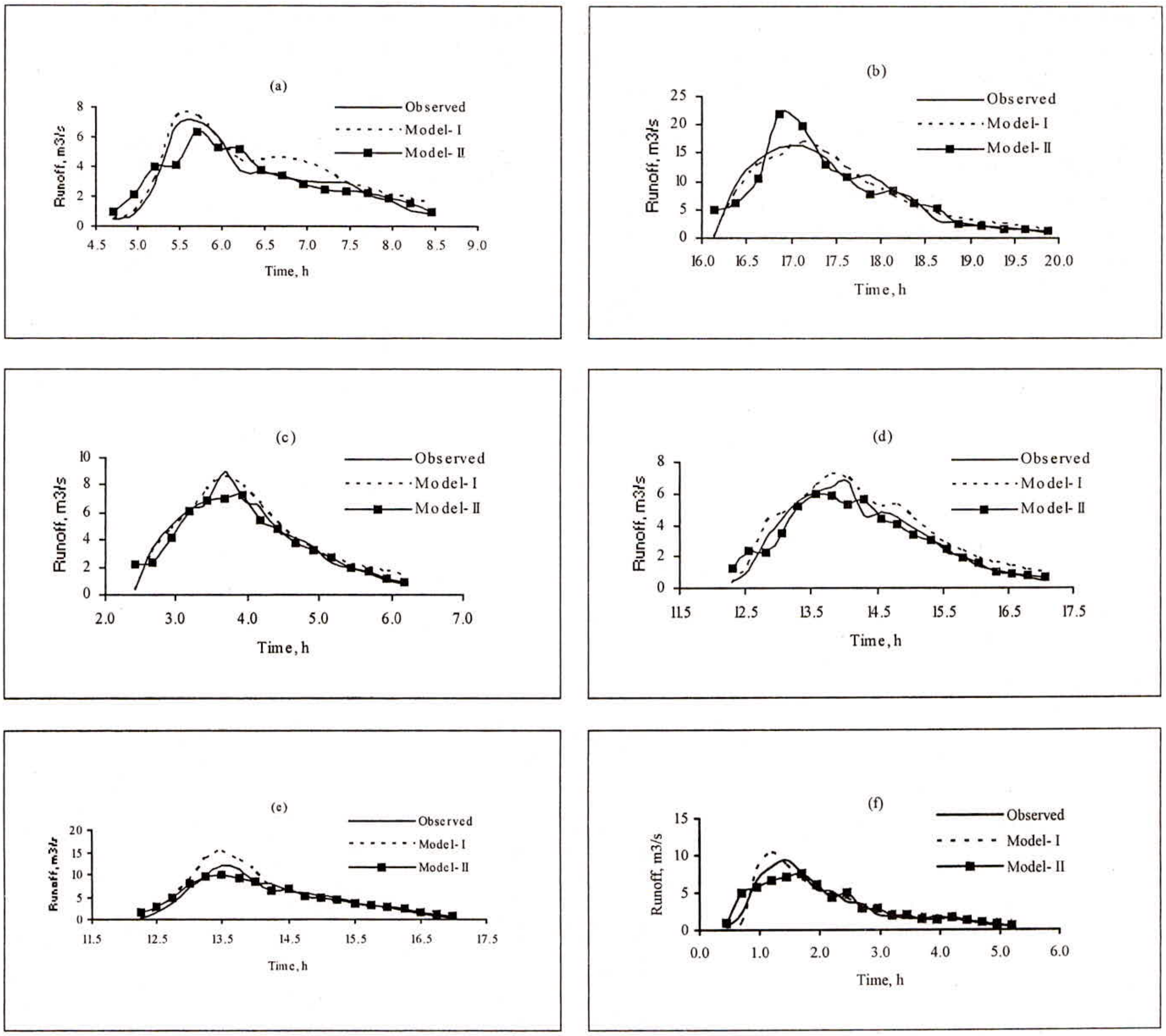

Fig. 2: Runoff Hydrographs for Different Storm Events: (a) 5.9.1993; (b) 28.6.1994; (c) 7.3.1995; (d) 10.2.1996; (e) 8.7.1997; (f) 5.8.1997

Table 2: Statistical Indices of Developed Models for the Study Area

Date of Storm	*RTIC*		*ISE, %*		*CC*		*CE, %*	
	Model I	*Model II*	*Model I*	*Model II*	*Model I*	*Model II*	*Model I*	*Model II*
Sept 5, 1993	0.38	0.58	5.44	8.20	0.976	0.87	88.43	73.69
June 28, 1994	0.38	1.06	3.53	11.18	0.982	0.91	96.42	81.45
March 7, 1995	0.23	0.39	2.88	4.34	0.986	0.91	98.72	89.20
Feb. 10, 1996	0.33	0.47	4.17	4.68	0.987	0.92	91.86	90.73
July 8, 1997	0.63	0.30	6.28	3.12	0.971	0.92	84.10	95.97
August 5, 1997	0.45	0.52	5.55	5.47	0.967	0.92	91.14	86.73

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