ICIWRM – 2000, Proceedings of International Conference on Integrated Water Resources
Management for Sustainable Development, 19 – 21 December, 2000, New Delhi, India

Evolution of water table with subsurface drainage for variable recharge and constant ET

SURJEET SINGH and SUSHIL KUMAR SINGH
G.P.S. Regional Centre, National Institute of Hydrology, Sagar (M.P.) – 470 001, India

Abstract

Subsurface drainage plays an important role in lowering the water table in case of excess irrigation and recharge due to rainfall. The shallow groundwater table affected by the recharge in the presence of drainage is required for the design of subsurface drains and management of crop practices. Another important factor affecting the shallow groundwater table is the evapotranspiration or evaporation from groundwater table. Many analytical studies have been reported in the literature for the evolution of water table with subsurface drainage. In most of the studies, constant recharge has been considered. Only in few studies variable recharge has been considered but without considering evapotranspiration (ET).

In the present study, the analytical solution has been proposed for the combination of variable recharge functions and constant rate evapotranspiration. Linear and exponential variation of recharge with time have been considered. For obtaining the solution the Boussinesq equation has been linearized by suitable substitution and the approaches applicable for the linear system have been adopted. The solutions, thus obtained have been shown to converge to the existing solution under specific conditions. The results have been presented to show the effects of variation in recharge and ET on the evolution of the water table.

INTORDUCTION

Irrigation is practised in many parts of arid and semi-arid regions of the world to enhance the agriculture production. Under such conditions seepage from canal beds, irrigated lands and other sources leads to the build-up of water table near to the ground surface and causes water logging and salinity problems in the top productive root zone. The recharge to groundwater may also occur due to natural rainfall, return flow, or impounded water. Besides recharge, a significant factor affecting the shallow groundwater table is evapotranspiration (ET). The problems of poor drainage can be alleviated by implementation of proper drainage systems for which a better understanding of the spatio-temporal variable of the water table in response to recharge and ET is very essential.

The problem of water table fluctuation in response to recharge and ET is of interest to hydrologists. Many researchers (Maasland, 1959; Schmid and Luthin, 1964) have studied the water table rise for steady state conditions. Most of the available analytical solutions have been developed assuming constant rate of recharge (Glover, 1961; Dagan, 1967; Hantush, 1967; Rao and Sarma, 1981; Ram and Chauhan, 1987). However, theoretical studies and field experiments have confirmed that the recharge which results from infiltration is time dependent (Morel-Seytoux, 1984;). The linear or exponential recharge rate were considered by Ram and Chauhan, 1987; Singh et al., 1991; and Ramana et al., 1995, but without considering ET. Skaggs (1975) has reported a numerical model of the Boussinesq equation for the water table affected by ET.

In this paper, analytical solutions are derived for the cases of linear or exponential recharge with constant ET using a transformation technique. The solution is compared with published solutions under specific condition of recharge and ET.

MATHEMATICAL FORMULATION

Figure 1 shows a schematic view of the problem. Initially the water table is at h_o height above the drain level. It rises due to recharge and declines as a result of ET. The drains are placed at a distance L apart. The aquifer is assumed to be homogeneous and isotropic with horizontal base.

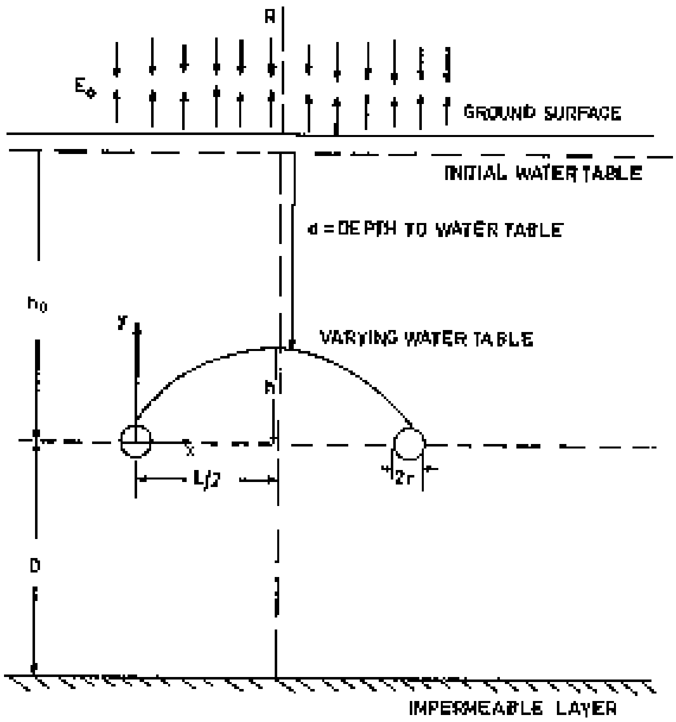

Figure 1. Schematic view of the problem.

The linearized Boussinesq equation (1904) for the transient groundwater flow in one dimension describing the flow system may be written as

$$K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) = f\,\frac{\partial h}{\partial t} \qquad (1)$$

where K = hydraulic conductivity, (LT^{-1}); h = height of the water table above the drains at mid-spacing, (L); f = drainable porosity, ($L^3\,L^{-3}$); x = horizontal space co-ordinate, (L);

and t = time, (T). When the water table is affected by recharge and evapotranspiration, equation 1 can be written as

$$K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+R(t)-E_o = f\,\frac{\partial h}{\partial t} \tag{2}$$

where R(t) = rate of recharge expressed as a function of time, (LT^{-1}); and E_o = constant ET, (LT^{-1}). Since equation 2 is in non-linear form, its analytical solution is difficult to obtain. However, the analytical solution can be obtained for linearized form of equation 2. Since 'h' is small in comparison to the depth above the impervious layer D, the term $\left(\frac{\partial h}{\partial x}\right)^2$ can be neglected and the variable 'h' appearing with $\left(\frac{\partial^2 h}{\partial x^2}\right)$ can be replaced with constant D. With this assumption, the equation 2 reduces to the following linearized form

$$KD\frac{\partial^2 h}{\partial x^2}+R(t)-E_o = f\,\frac{\partial h}{\partial t} \tag{3}$$

where, $D = d_e + h_o/2$ = average effective thickness of the saturated zone, (L); d_e = equivalent depth, (L); and h_o = initial water table height above the drains, (L). The following initial and boundary conditions apply:

$$h(x,0) = h_o \qquad t \le 0 \;,\; 0 \le x \le L \tag{4}$$
$$h(0,t) = 0 \qquad t > 0 \tag{5}$$
$$h(L,t) = 0 \qquad t > 0 \tag{6}$$

ANALYTICAL SOLUTIONS

Drainage with Linear Recharge and Constant ET

The analytical solution for the linear recharge and constant ET has been obtained for the recharge function, $R(t) = r_o + r\,t$, where, r_o = constant recharge, (LT^{-1}); r = coefficient of linear recharge, (LT^{-2}); and t = time, (T). Incorporating the value of linear recharge function in equation 3, the boundary value problem for the linearly varying recharge case is written as

$$\frac{KD}{f}\frac{\partial^2 h}{\partial x^2}+\frac{r_o+rt}{f}-\frac{E_o}{f}=\frac{\partial h}{\partial t} \tag{7}$$

For solving equation 7, the following transformation is devised

$$h(x,t)=z(x,t)+\frac{rt^2}{2f}+\left(\frac{r_o-E_o}{f}\right)t \tag{8}$$

This transformation transforms equation 7 into the following standard (heat flow equation) form.

$$\frac{\partial^2 z}{\partial x^2}=\frac{f}{KD}\frac{\partial z}{\partial t} \tag{9}$$

Using the same transformation, the initial and boundary conditions can be transformed as

$z(x, 0) = h_o \quad , \quad 0 \le x \le L \quad , \quad t \le 0$ (10)

$z(0, t) = -\dfrac{rt^2}{2f} - \left(\dfrac{r_o - E_o}{f}\right)t \quad , \quad t > 0$ (11)

$z(L, t) = z(0, t) \quad , \quad t > 0$ (12)

The solution of equation 9 with initial and boundary conditions given by equations 10 to 12 is obtained using the solution given by Carslaw and Jaeger (1959). The solution of equation 7 is then obtained by taking the inverse transformation given by equation 8. The final solution is given below:

$$h(x,t) = \frac{4}{\pi}\sum_{n=1,3,5,..}^{\infty}\left[\frac{h_o e^{-n^2\alpha t}}{n} + \frac{r}{f\alpha n^3}\left\{t - \frac{(1-e^{-n^2\alpha t})}{n^2\alpha}\right\} + \frac{(r_o - E_o)(1-e^{-n^2\alpha t})}{f\alpha n^3}\right]\sin\left(\frac{n\pi x}{L}\right) \quad (13)$$

where $\dfrac{\pi^2 KD}{fL^2} =$ the reaction factor (α), (T^{-1})

Special Cases

Linear Recharge Only: Equation 13 for the value of $E_o = 0$, $r_o = 0$ and $h_o = 0$ takes the following form:

$$h(x,t) = \frac{4rL^2}{\pi^3 KD}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^3}\left[t - \frac{(1-e^{-n^2\alpha t})}{n^2\alpha}\right]\sin\left(\frac{n\pi x}{L}\right) \quad (14)$$

This equation may be compared with the solution given by Ram and Chauhan (1987) for the linear recharge case.

Constant Recharge Only: Equation 13 for the value of $E_o = 0$, $r = 0$ and $h_o = 0$ takes the following form:

$$h(x,t) = \frac{4r_o L^2}{\pi^3 KD}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^3}(1-e^{-n^2\alpha t})\sin\left(\frac{n\pi x}{L}\right) \quad (15)$$

This equation is similar to the solution provided by Maasland (1959) for constant recharge case.

Constant ET Only: Equation 13 for the value of $r = 0$ and $r_o = 0$ takes the following form:

$$h(x,t) = \frac{4}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n}\left[h_o e^{-n^2\alpha t} - \frac{E_o(1-e^{-n^2\alpha t})}{f\alpha n^2}\right]\sin\left(\frac{n\pi x}{L}\right) \quad (16)$$

This equation may be compared with the solution given by Singh et al. (1996) in presence of constant ET.

No Recharge and No ET: Equation 13 for the value of $r = 0$, $r_o = 0$ and $E_o = 0$ takes the following form:

$$h(x,t) = \frac{4h_o}{\pi} \sum_{n=1,3,5,..}^{\infty} \frac{e^{-n^2 \alpha t}}{n} \sin\left(\frac{n\pi x}{L}\right) \tag{17}$$

This equation is mathematically similar to the solution given by Glover (1954).

Drainage with Exponentially varying Recharge and Constant ET

The analytical solution for exponentially varying recharge with constant ET has been obtained for the recharge function, R(t) = $ce^{-\beta t}$, where c = initial rate of recharge, (LT^{-1}); β = decay constant, (T^{-1}). Incorporating the value of exponential recharge function in equation 3, the boundary value problem for the linearly varying recharge case is written as

$$\frac{KD}{f} \frac{\partial^2 h}{\partial x^2} + \frac{ce^{-\beta t}}{f} - \frac{E_o}{f} = \frac{\partial h}{\partial t} \tag{18}$$

For solving equation 18, the following transformation is devised.

$$h(x,t) = z(x,t) - \frac{c}{\beta f} e^{-\beta t} - \frac{E_o}{f} t \tag{19}$$

This transformation transforms the equation 18 into the standard form given by equation 9. Using the same transformation, the initial and boundary conditions can be transformed as

$$z(x, 0) = h_o \;, \quad 0 \le x \le L \;, \quad t \le 0 \tag{20}$$

$$z(0, t) = \frac{c}{\beta f} e^{-\beta t} + \frac{E_o}{f} t \;, \; t > 0 \tag{21}$$

$$z(L, t) = z(0, t) \quad , \quad t > 0 \tag{22}$$

The solution of equation 18 with initial and boundary conditions given by equations 4 to 6 is obtained as

$$h(x,t) = \frac{4}{\pi} \sum_{n=1,3,5,..}^{\infty} \left[\frac{h_o e^{-n^2 \alpha t}}{n} - \frac{ce^{-\beta t}}{n\beta f} \left\{ 1 - \frac{(1 - e^{-(n^2\alpha - \beta)t})}{(1 - \frac{\beta}{n^2 \alpha})} \right\} - \frac{E_o (1 - e^{-n^2 \alpha t})}{f \alpha n^3} \right] \sin\left(\frac{n\pi x}{L}\right) \tag{23}$$

Special Cases

Exponential Recharge Only: Equation 23 for the value of E_o = 0 with h_o = 0 reduces to the following form:

$$h(x,t) = \frac{4ce^{-\beta t}}{f\beta\pi} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n} \left\{ \frac{(1 - e^{-(n^2\alpha - \beta)t})}{(1 - \frac{\beta}{n^2 \alpha})} - 1 \right\} \sin\left(\frac{n\pi x}{L}\right) \tag{24}$$

This equation may be compared with the solution given by Ram and Chauhan (1987) for the exponentially varying recharge case.

Constant ET Only: Equation 23 for the value of c = 0 reduces to the equation 16.

No Recharge and No ET: Equation 23 for the value of c = 0 and E_o = 0 reduces to the equation 17 which is similar to the solution given by Glover (1954).

RESULTS AND DISCUSSION

To test the performance of the analytical solutions for a physical situation, a subsurface drainage system is considered with tile drains placed at 1.75 m below the soil surface and at a spacing of 50 m apart. The soil parameters for this purpose have been taken from Singh et al. (1996). The hydraulic conductivity of soil is taken as 0.08 m/day and drainable porosity as 0.10. The average effective thickness of the saturated zone is 3.50 m. Singh et al. (1996) proposed an analytical solution which takes into account the effect of variable ET as a function of water table depth. For the comparison, equation 13 is used to calculate the water table decline. The results of the proposed solution and Singh et al. (1996) solution are presented in Table 1. It may be seen from the table that in both the solutions trend of water table decay is similar and the results are comparable.

Table 1. Ground water table behaviour in presence of constant ET.

(E_o = 0.008 m/day)

Time (days)	Dimensionless Water Table Height (h/h_o)	
	Singh et al. (1996)	Proposed Solution
2	0.8731	0.8729
4	0.6390	0.6381
6	0.4176	0.4158
8	0.2354	0.2329
10	0.0888	0.0856
12	-0.0289*	-0.0326*

- The value is negative because the ET continues even after the water table has been lowered to drain depth.

Effect of Linear Recharge on the Water Table Height

Table 2 shows that the water table height increases with increase in recharge as a function of time. For small value of recharge water table is declining. However, with increase in the value of recharge, the water table initially declines and then rises. This rise in water table occurs as a consequence of higher rate of recharge with increasing time.

Effect of Exponential Recharge on the Water Table Height

The effect of exponential recharge is presented in Table 3 with and without consideration of constant ET. The exponential recharge pattern $R(t) = ce^{-\beta t}$ is taken to estimate the midpoint water table height. It can be seen from the table that in beginning the water table rises at a faster rate. But on the third day it starts decaying with time. This decay of water table is only because of decrease in the rate of recharge with time.

Table 2. Effect of linear recharge on the water table height.

Time	Dimensionless Water Table Height (h/h_o)		
(days)	r = 0.001	R = 0.003	r = 0.006
0	1.0000	1.0000	1.0000
2	0.9751	0.9979	1.0320
4	0.8548	0.9440	1.0776
6	0.7512	0.9437	1.2325
8	0.6887	1.0145	1.5034
10	0.6632	1.1467	1.8719
12	0.6681	1.3286	2.3192

Table 3. Effect of exponential recharge on the water table.

(c = 0.0371 and β = 0.571)

Time	Dimensionless Water Table Height (h/h_o)	
(days)	E_o = 0.0000	E_o = 0.008 m/day
0	0.6311	0.6311
2	0.8559	0.7652
4	0.8107	0.6386
6	0.6889	0.4499
8	0.5646	0.2718
10	0.4565	0.1206
12	0.3672	-0.0033*

- The value is negative because the ET continues even after the water table has been lowered to drain depth.

Table 4. Effect of ET on the Water Table Decline.

Time	Dimensionless Water Table Height (h/h_o)	
(days)	E_o = 0.0000	E_o = 0.008 m/day
0	1.0000	1.0000
2	0.9637	0.8729
4	0.8103	0.6381
6	0.6549	0.4158
8	0.5257	0.2329
10	0.4215	0.0856
12	0.3379	-0.0326*

***** The value is negative because the ET continues even after the water table has been lowered to drain depth.

Effect of Constant ET on the Water Table

To show the effect of ET on the decline of water table height, calculations are done using equation 13. The value of constant ET is taken as 0.008 m/day which represents the potential ET. The dimensionless water table height (h/h_o) versus time is shown in Table 4. It is observed that water table is significantly affected by ET. It is observed that the water table declines at a faster rate if ET is considered to be occurring at constant rate and declines still faster if ET is affected by the falling water table. With the consideration of ET, the fast decline of water table or the decrease in time to lower the water table could be translated into increased drain spacing.

CONCLUSIONS

Analytical solutions have been derived for water table evolution with subsurface drainage under linearly and exponentially varying rate of recharge with constant evapotranspiration (ET). The solutions have been shown to converge to available solutions under specific conditions for recharge, ET and boundary. Effect of variable recharge rate and ET on water table have been presented and discussed.

Acknowledgement

The authors are thankful to Dr. H.S. Chauhan, Professor Emeritus (AICTE), G.B. Pant Univ. of Agr. and Tech., Pantnagar for suggestions and help. The permission of Director, National Institute of Hydrology, Roorkee, to publish this paper is thankfully acknowledged.

References

Boussinesq, J.(1904), "Recherches theorretiques sur l' ecoulement des nappes d' eau infiltrees dans le sol et sur le debit des sources", Z. Agnew. Math. Phys.., 10 pp: 5-78.

Dagan, G.(1967), "Linearized solutions of free surface ground water flow with uniform recharge", J. Geophys. Res. 72(4) pp:1183-1193.

Glover, R.E.(1961), "Mathematical derivations as pertain to groundwater recharge", Agr Res. Serv., USDA, Ft. Collins, Colo., 81 pp (mimeo).

Hantush, M.S.(1967), "Growth and decay of ground water mounds in response to uniform percolation", Water Resour. Res. 3(1) pp:227-234.

Maasland, M., (1959). Water table fluctuation induced by intermittent recharge. Jour. of Geophy. Res., 64(5) : 549-559.

Morel-Seytoux, H.J.(1984), "From excess infiltration to aquifer recharge: A derivation based on the theory of flow of water in unsaturated soils", Water Resour. Res. 20(9) pp: 1230-1240.

Ram, S. and Chauhan, H.S.(1987), "Analytical and experimental solutions for drainage of sloping lands with time varying recharge", Water Resour. Res. 23(6) pp: 1090-1095.

Ram, S. and Chauhan, H.S.(1987), "Drainage of sloping lands with constant replenishment", J. of Irrig. and Drainage Engrg, 113(2) pp: 213-223.

Ramana, D.V., Rai, S.N. and Singh, R.N.(1995), "Water table fluctuation due to transient recharge in a 2-D aquifer system with inclined base", Water Resour. Management 9 pp: 127-138.

Rao, N.H. and Sarma, P.B.S.(1981), "Recharge from rectangular areas to finite aquifers", J. Hydrol. 53:269-275.

Schmid, P. and Luthin, J.(1964), "The drainage of sloping lands, J. Geophys. Res., 69: 1525-529.

Skaggs, R.W.(1975), "Drawdown solutions for simultaneous drainage and ET", J. of the Irrig. and Drainage Div., Proc. of ASCE 101(IR-4) pp: 1279-1291.

Singh, R.K., Parasher, S.O., Chauhan, H.S., Gupta, S.K., Bonnell, R.B. and Madramootoo, C.A.(1996), "An analytical solution of the Boussinesq equation for subsurface drainage in the presence of evapotranspiration", Trans. ASAE 39(3) pp: 953-960.

Singh, R.N., Rai, S.N. and Ramana, D.V.(1991), "Water table fluctuations in a sloping aquifer with transient recharge", J. Hydrol. pp: 315-326.