APPENDIX—II

NASH MODEL

II.1. The Linear Differential Equation with Constant Coefficient

With the assumption that the input and output are related by the linear differential equation with constant coefficients, the equation may be written as :

$$x\ (t) = a \frac{d^n}{dt^n} y\ (t) + b \frac{d^{n-1}}{dt^{n-1}} y\ (t) + c \frac{d^{n-2}}{dt^{n-2}} y\ (t) + \qquad (II.1)$$

$$x\ (t) = (a\ D^n + b\ D^{d-2} + C\ D^{n-2} + \ldots\ldots\ldots)\ y\ (t) \qquad (II.2)$$

here, x (t) is input and y (t) is output.

As x (t) is usually known and y (t) is unknown the Eq. (II.2) may be written as :

$$y\ (t) = \frac{1}{(aD^n + bD^{n-1} + c\ D^{n-2} \ldots\ldots\ldots)} x\ (t) \qquad (II.3)$$

which may be factorised as :

$$y\ (t) \frac{A}{(C_1 + D)\ (C_2 + D)\ (C_3 + D)\ldots} x\ (t) \qquad [II.4)$$

We may assume that all the elements in the catchment operate in a stable and highly damped manner, i e. the effect of a disturbance such as instantaneous rainfall dies out eventually without oscillating about zero. In such cases the roots of the polynomial are all real and negative (Eq. II.4) i.e. the C's in Eq. (II.4) are all real and positive. It may also be shown that if the requirement of continuity is observed (i.e. if $\int_0^\infty i\ dt = \int_0^\infty qdt)$ as it is in the present case then

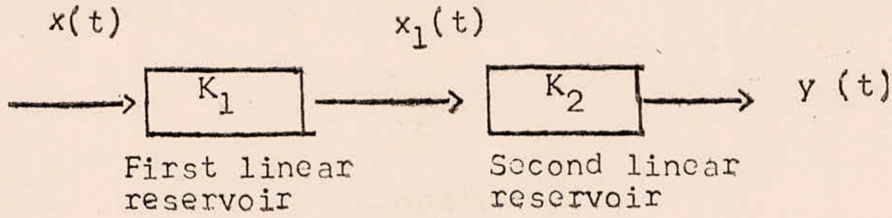
$$A = C_1, C_2. C_3, \ldots\ldots$$

Therefore Eq. [II.4] may be written as :

$$y\ (t) = \frac{1}{(1 + K_2D)} \quad \frac{1}{(1 + K_2D)} \quad \frac{1}{(1 + K_2D)} \ldots\ldots i\ (t) \qquad (II\ 5)$$

where, K's are all real and positive

The Eq. (II.5) is the general linear differential equation of a stable highly damped system which obeys the law of continuity and whose elements do not change with time. Now consider a linear reservoir with storage coefficient K. As we know for a linear reservoir the storage S is proportional to y i e. S = Ky here, y is outflow from the linear reservoir.



and hence :

$$S(t) = Ky(t) \tag{II.6}$$

Equation of continuity tells us that inflow-outflow = rate of change of storage :

$$\text{i.e. } x(t) - y(t) = \frac{dS(t)}{dt} \tag{II.7}$$

From equation (II.6) $\frac{dS(t)}{dt} = K\frac{dy(t)}{dt}$ (II.8)

Therefore, $x(t) - y(t) = K\frac{dy(t)}{dt}$ (II.9)

Denoting $\frac{d}{dt}$ as a differential operator D, the Eq. (II.9) may be written as :

$$x(t) - y(t) = K\,Dy(t) \tag{II.10}$$

or $\quad KDy(t) + y(t) = x(t)$ (II.11)

or $\quad (1 + KD)\,y(t) = x(t)$ (II.12)

or $\quad y(t) = \frac{1}{(1+KD)}\,x(t)$ (II.13)

Eq. (II.13) shows a relationship between input and output with an operator $\frac{1}{(1+KD)}$ for single linear reservoir.

Now we consider the two linear reservoirs in series and their storage coefficients are K_1 and K_2. Since the two reservoirs are in series, the outflow from the first reservoir will be inflow in the second reservoir :

Taking first linear reservoir into consideration, the operation of the first linear reservoir may be written according to Eq. (II.13).

$$\text{i.e. } x_1(t) = \frac{1}{1 + K_1 D}\,x(t) \tag{II.14}$$

Similarly, the operation of the second linear reservoir may be expressed as

$$y(t) = \frac{1}{1 + K_2 D}\,x_1(t) \tag{II.15}$$

Substituting value of x_1 (t) from Eq. (II.14) in Eq. (II.15), we will get : (for linear reservoirs)

$$y(t) = \frac{1}{(1+K_1 D)} \frac{1}{(1+K_2 D)} \cdots \frac{1}{(1+K_n D)} x(t) \tag{II.16}$$

Here, K's are storage coefficients for n linear unequal reservoirs connected into series. If we want to know impulse response of the system, we have to know (n + 1) parameters i.e, values of K and value of n itself. Method of moments is one of the most important methods and easy to handle from mathematical point of view. If input and output data is in error, the higher order moments will be associated with magnified error. Since to estimate (n+1) parameters, (n+1) moments are required which may not give reliable estimate of parameters due to magnification in error for higher order moments. The easiest way is to reduce number of parameters taking n linear reservoir of equal storage coefficients and the operation of n equal linear reservoirs is given as :

$$y(t) = \left(\frac{1}{1+KD}\right)^n x(t) \tag{II.17}$$

Eq. (II.17) represents the system having two parameters n and K. Note that it is not necessary that the value of n will be always integer it may be real as well.

The Laplace transform of Eq. (II.17) is given as ;

$$y(s) = \frac{1}{(1+Ks)^n} x(s) \tag{II.18}$$

$$y(s) = U(s)\, x(s) \tag{II.19}$$

where, $$u(s) = \frac{1}{(1+Ks)^n} \tag{II.20}$$

Since inverse Laplace transform of the function u (s) given as :

$$u(t) = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{ds^{n-1}} u(s) e^{st} (s-p)^n\right] s = p \text{ for nth order pole.}$$

$$\text{Hence, } u(t) = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{ds^{n-1}} \left(\frac{1}{1+Ks}\right)^n e^{st} (s-p)^n\right] s=p \text{ for nth order pole} \tag{II.21}$$

$$= \frac{1}{(n-1)!} \left[\frac{ds^{n-1}}{ds^{n-1}} \left(\frac{1}{K\left(s+\frac{1}{K}\right)}\right)^n e^{st} (s-p)^n\right] s=p$$

$$= \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{ds^{n-4}} \frac{1}{K^n} \frac{1}{\left(s+\frac{1}{K}\right)^n} e^{st} (s-p)^n\right] s=p$$

Putting $p = -\frac{1}{K}$

$$\frac{1}{K} = -P$$

$$\left(\frac{1}{K}\right)^n = (-p)^n$$

$$= \left(\frac{1}{(n-1)\,!} \left[\frac{d^{n-1}}{ds^{n-1}} \quad \frac{(-p)}{(s-p)^n} e^{st} (s-p)^n\right] s=p\right.$$

$$= \frac{(-p)^n}{(n-1)\,!} (t)^{n-1} e^{pt}$$

$$u(t) = \frac{1}{K^n (n-1)!} \quad (t)^{n-1} e^{-t/K}$$

$$= \frac{1}{K(n-1)\,!} \quad \left(\frac{t}{K}\right)^{n-1} e^{-t/K} \tag{II.22}$$

So impulse response or IUH for the system having n equal linear reservoirs of storage coefficient K is given by the Eq. (II.22).

II.2. First and second moments of impulse response about the origin

The equation of rth moment of any function f (t) about the origin is given as ;

$$_rM'_f = \int_{-\infty}^{\infty} f(t)\ (t)^r\ dt \tag{II.23}$$

Therefore rth moment of impulse response u (t) is given by the relation

$$_r{}'u = \int_{-\infty}^{\infty} u(t)\ (t)^r\ dt \tag{II.24}$$

Since the system is initially at rest, therefore, the lower limit of Eq (II.24) will be equal to zero and hence :

$$_rM'u = \int_{-\infty}^{\infty} \frac{1}{K(n-1)\,!} \quad \left(\frac{t}{K}\right)^{n-1} e^{-t/K} (t)^r\ dt$$

putting $t/K = x$

$$_rM'u = \int_{0}^{\infty} \frac{1}{(n-1)\,!} x^{n-1} e^{-x} t^r\ dx$$

for first moment about the origin $r = 1$

$$1\ M'u = \int_{0}^{\infty} \frac{1}{(n-1)\,!} x^{n-1} e^{-x}\ t\ dx$$

$$= K \int_{0}^{\infty} \frac{1}{(n-1)\,!} x^{n-1} e^{-x} \left(\frac{t}{K}\right) dx$$

$$= K \int_{0}^{\infty} \frac{1}{(n-1)!} x^{n} e^{-x}\ dx$$

$$= \frac{K}{(n-1)!} \int_0^\infty x^n e^{-x} dx$$

$$= \frac{K}{(n-1)!} \lceil \overline{n+1}$$

For real value of n, $(n-1)! = \lceil \overline{n}$

$$\therefore {}_1M'_u = \frac{K}{\lceil \overline{n}}, \; \lceil \overline{n+1} = Kn$$

$${}_1M'_u = nK \qquad (II.25)$$

for second moment about the origin $r = 2$.

$${}_2M'_u = \frac{1}{(n-1)!} \int_0^\infty x^{n-1} e^{-x} t^2 dx$$

$$= \frac{K^2}{\lceil \overline{n}} \int_0^\infty x^{n-1} e^{-x} (t/K)^2 dx$$

$$= \frac{K^2}{\lceil \overline{n}} \int_0^\infty x^{n-1} x^2 e^{n-x} dx$$

$$= \frac{K^2}{\lceil \overline{n}} \int_0^\infty x^{n+1} e^{-x} dx$$

$$= \frac{K^2}{\lceil \overline{n}} \lceil \overline{n+2} = \frac{K^2}{\lceil \overline{n}} (n+1) \lceil \overline{n+1} = \frac{K^2}{\lceil \overline{n}} n (n+1) \lceil \overline{n}$$

$$= n K^2 (n+1)$$

$${}_2M'_u = (nK)^2 + nK^2 \qquad (II.26)$$

Now theorem of moments introduced by Nash (1959), may be used to relate moments of input and output with the moments of impulse response. Using moments theorem. (See APPENDIX—III for the proof).

$${}_rM'y = (M'_x + M'_u)^r \qquad (II.27)$$

where, suffixes are written as power indices without of course interpreting them as such except for purposes of expansion.

Expanding Eq. (II.27) for $r = 1$ and $r = 2$, the resulting equations are :

$${}_1M'y = {}_1M'x + {}_1M'u \qquad (II.28)$$

$${}_2M'y = {}_2M'x + 2\,{}_1M'x\,{}_1M'u + {}_2M'u \qquad (II.29)$$

Eq. (II.28) and (II.29) may be written as :

$$_1M'u = {_1M'}y - {_1M'}x \quad \text{(II.30)}$$

$$_2M'u = {_2M'}y - {_2M'}x - 2\,{_1M'}x \,.\, {_1M'}u \quad \text{(II.31)}$$

Combining Eq. (II.25) and (II.30), Eq. (II.26) and (II.31), we get :

$$_1M'y - {_1M'}x = nK \quad \text{(II.32)}$$

$$_2M'y - {_2M'}x - 2 \,.\, {_1M'}x \,.\, {_1M'}u = (nK)^2 + nK^2$$

$$_2M'y - {_2M'}x = (nK)^2 + nK^2\ 2\,nK \,.\, {_1M'}x$$

$$_2M'y - {_2M'}x = n\,(n + 1)\,K^2 + 2nK \,.\, {_1M'}x \quad \text{(II 33)}$$

Since moment of IUH is normalized, so moments of input and output should also be normalised before using in the above Eq. (II.32) and (II.33). In the case of calculating moments, excess rainfall is assumed as input and direct surface runoff as output.

II.3 Unit Hydrograph from Impulse response (IUH)

Since the unit hydrograph of period T is expressed in terms of S-curve hydrograph by the equation :

$$u\,(T,\,t) = \frac{1}{T}\,[S\,(t) - S\,(t - T)] \quad \text{(II.34)}$$

where, S (t) is S-curve ordinates of unit intensity and S (t — T) is the ordinates of the S-curve hydrograph but shifted by T hours.

The general equation for the IUH of Nash model is given as :

$$u\,(t) = \frac{1}{K\,\overline{|n}}\,e^{-t/K}\,(t/K)^{n-1} \quad \text{(II.35)}$$

As discussed in the lecture no 6 on 'Unit Hydrograph Analysis' that relationship between the S-curve and IUH is given by the equation :

$$S\,(t) = \int_{O}^{t} u\,(o,t)\,dt \quad \text{(II.36)}$$

therefore,

$$S\,(t) = \frac{1}{\overline{|n}} \int_{O}^{t/K} e^{-t/K}\,(t/K)^{n-1}\,d\,(t/K) \quad \text{(II 37)}$$

$$= I\,(n,\,t/K) \quad \text{(II.38)}$$

where, I (n, t/K) is the incomplete gama function of order n at (t/K).
Similarly,

$$S\,(t-T) = I\left(n,\,\frac{t-T}{K}\right)$$

Hence from Eq. (II.37)

$$U\,(T,\,t) = \frac{1}{T}\left[\,I\left(n,\,\frac{t}{K}\right) - I\left(n,\,\frac{t-T}{K}\right)\right] \quad \text{(II.39)}$$

which is the general equation of the unit hydrograph of period T hour.