LECTURE -5

MUSKINGUM-CUNGE METHOD

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Objective: The objective of this lecture is to describe the methodology for stream channel routing using Muskingum-Cunge method. The advantages and limitations of the Muskingum-Cunge method are also highlighted in the lecture.

1.0 INTRODUCTION

As discussed in one of the earlier lectures, the routing equation for conventional Muskingum method is given as:

$$O_{n+1} = C_o I_{n+1} + C_1 I_n + C_2 O_n \qquad \ldots.(1)$$

in which, C_o, C_1 and C_2 are routing coefficients defined in terms of Δt, K, and X as follows:

$$C_o = \frac{(\Delta t/K) - 2X}{2(1-X) + (\Delta t/K)} \qquad \ldots.(2)$$

$$C_1 = \frac{(\Delta t/K) + 2X}{2(1-X) + (\Delta t/K)} \qquad \ldots.(3)$$

$$C_2 = \frac{2(1-X) - (\Delta t/K)}{2(1-X) + (\Delta t/K)} \qquad \ldots.(4)$$

where, I = Inflow, O = Outflow, K = a time constant or storage coefficient and X = a dimensionless weighting factor.

The Muskingum method can calculate runoff diffusion, ostensibly by varying the parameter X. A numerical solution of the linear kinematic wave equation using a third order - accurate scheme (current number C = 1) leads to pure flood hydrograph translation (see lecture on 'Hydraulic Methods of Flood Routing'). Other numerical solutions to the linear kinematic wave equation invariably produce a certain amount of numerical diffusion and/or dispersion. The Muskingum and linear kinematic wave routingequation are strikingly similar. Further, unlike the kinematic wave equation, the diffusion wave equation does have the capability to describe the physical diffusion.

From these propositions, Cunge (1969) concluded that the Muskingum method is essentially a linear kinematic wave solution and that the flood wave attenuation shown by the calculation is due to the numerical diffusion of the scheme itself. He discretized the kinematic wave equation on the xt plane (Fig.1) in wave that parallels the Muskingum method to prove this assertion and came out with a physically based alternative to the Muskingum method. The alternative method is popularly known as Muskingum-Cunge method.

2.0 MUSKINGUM CUNGE METHOD

The kinematic wave equation discussed in lecture on 'Hydraulic Methods of Flood Routing' is given as:

$$\frac{\partial Q}{\partial t} + c\,\frac{\partial Q}{\partial x} = 0.0 \qquad (5)$$

in which, $c = \beta v$ is the kinematic wave celerity.

Eq.(5) was discretized by Cunge (1969) on the xt plane shown in Fig. 1 in a way that parallels the Muskingum method, wherein the spatial derivative was centred and the temporal derivative was off centered by means of a weighting factor X. The resulting equation is given as:

$$\frac{X\,(Q_j^{n+1} - Q_j^{n}) + (1-x)\,(Q_{j+1}^{n+1} - Q_{j+1}^{n})}{\Delta t}$$

$$+c\,\frac{(Q_{j+1}^{n} - Q_j^{n}) + (Q_{j+1}^{n+1} - Q_j^{n+1})}{2\Delta x} = 0 \qquad \ldots.(6)$$

Solving Eq.(6) for the unknown discharge leads to the following equation:

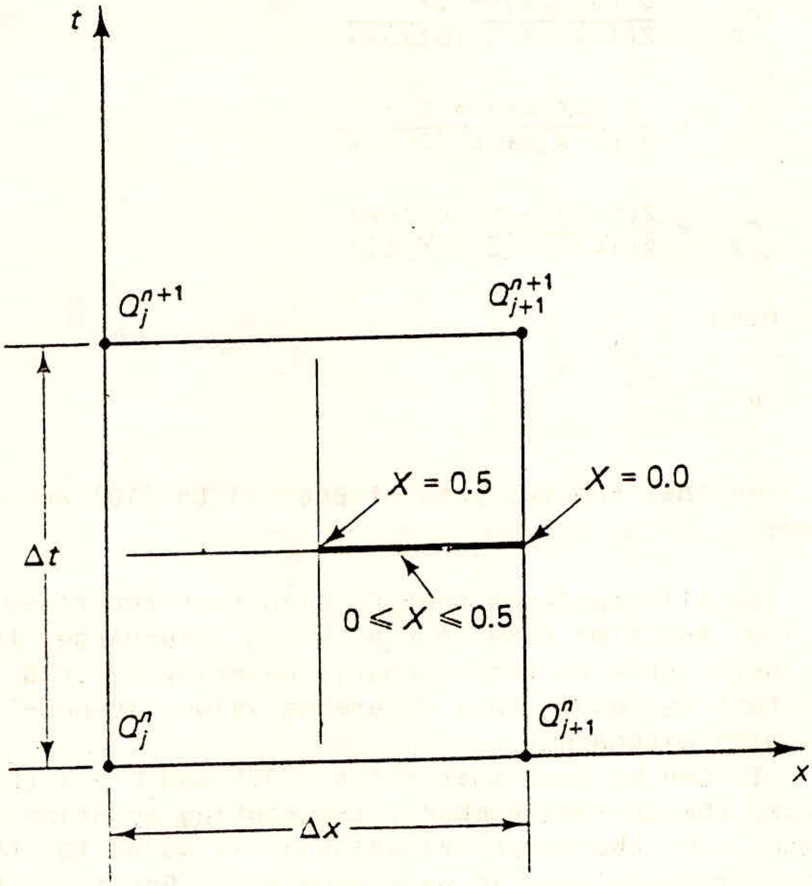

Figure 1. Space-time discretization of kinematic wave equation paralleling Muskingum method.

$$Q_{j+1}^{n+1} = C_o\, Q_j^{n+1} + C_1\, Q_j^n + C_2\, Q_{j+1}^n \qquad ....(7)$$

The routing coefficients are:

$$C_o = \frac{c\,(\Delta t/\Delta x) - 2X}{2(1-X) + c\,(\Delta t/\Delta x)} \qquad ....(8)$$

$$C_1 = \frac{c\,(\Delta t/\Delta x) + 2\,X}{2\,(1-X) + c\,(\Delta t/\Delta x)} \qquad ....(9)$$

$$C_2 = \frac{2(1-X) - c\,(\Delta t/\Delta x)}{2(1-X) + c\,(\Delta t/\Delta x)} \qquad ....(10)$$

By defining:

$$K = \frac{\Delta x}{c} \qquad ....(11)$$

it is seen that the two sets of Eqs.(8) to (10) and (2) to (4) are the same.

Eq.(11) confirms that K is in fact the flood wave travel time, i.e. the time taken for a given discharge to travel the reach length Δx with the kinematic celerity c. In a linear mode, c is constant and equal to a reference value, in non-linear mode, it varies with discharge.

It can be seen that for $X = 0.5$ and $C = 1$ ($C = c\ \Delta t/\Delta x = \beta v\ \Delta t/\Delta x$, the courant number), the routing equation is third order accurate, i.e. the numerical solution is equal to the analytical solution of the kinematic wave equation. For $X = 0.5$ and $C \neq 1$, it is second order accurate, exhibiting only numerical dispersion. For $X \neq 0.5$ and $C \neq 1$, it is first order accurate exhibiting both numerical diffusion and dispersion. For $X \neq 0.5$ and $C = 1$, it is first order accurate, exhibiting only numerical diffusion. These relations are summarized in Table 1.

Table 1 : Numerical Properties of Muskingum Cunge Method

Parameter X	Parameter C	Order of Accuracy	Numerical Diffusion	Numerical Dispersion
0.5	1	Third	No	No

0.5	≠1	Second	No	No
<0.5	≠1	First	Yes	Yes
<0.5	1	First	Yes	No

In practice, the numerical diffusion can be used to simulate the physical diffusion of the coefficient (V_n) for the scheme can be derived by expanding the discrete function Q (j Δx, n Δt) in Taylor series about grid point (j Δx, n Δt) (Ponce, 1989):

$$V_n = c\,\Delta x\,(\tfrac{1}{2} - X) \qquad (12)$$

Eq.(12) reveals the following:

(i) For X = 0.5 there is no numerical diffusion, although there is numerical dispersion for C ≠ 1.

(ii) For X > 0.5, the numerical diffusion coefficient is negative, i.e. numerical amplification which explains the behaviour of the Muskingum method for this range of X values:

(iii) For Δx = 0, the numerical diffusion coefficient is zero, clearly the trival case.

The hydraulic diffusivity (V_h) which is a characteristics of flow and channel is defined as:

$$V_h = \frac{Q_o}{2TS_o} = \frac{q_o}{2S_o} \qquad \ldots.(13)$$

in which, $q_o = Q_o/T$ is the reference flow per unit of channel width.

A predictive equation for X can be obtained by matching the hydraulic diffusivity V_h (Eq.13) with the numerical diffusion coefficient of the Muskingum scheme (Eq.12). This leads to the following expression for X:

$$X = \frac{1}{2}\left(1 - \frac{q_o}{S_o c\Delta x}\right) \qquad \text{....(14)}$$

With X calculated by Eq. (14), the Muskingum method is referred to as Muskingum Cunge method (NERC, 1975). The routing parameter X can be calculated as a function of the following numerical and physical properties:

(i) Reach length Δx;

(ii) Reference discharge per unit width q_o;

(iii) Kinematic wave celarity, c, and

(iv) Bottom slope S_o.

It should be noted that the Eq.(14) was derived by matching physical and numerical diffusion (i.e. second order processes) and does not account for dispersion (a third order process). Therefore, in order to simulate flood wave diffusion properly with the Muskingum-Cunge method, it is necessary to optimise numerical diffusion with Eq.(14) while minimising numerical dispersion (by keeping the value of cuorant number as close to 1 as practicable).

A unique feature of the Muskingum method is the grid independence of the calculated out flow hydrograph. If numerical dispersion minimised (keeping cuorant number C close to one), the calculated outflow at the downstream end of a channel reach will be essentially the same regardless of how many sub-reaches are used in the computation. This is because X is a function of Δx and the routing co-efficients C_o, C_1 and C_2 vary with reach length.

An improved version of the Muskingum-Cunge method is due to Ponce and Yevjevich (1978). The current number, C is defined as the ratio of wave celerity (C) to grid celerity $\Delta x/\Delta t$ i.e.

$$C = c\,\frac{\Delta t}{\Delta x} \qquad \text{....(15)}$$

The grid diffusivity is defined as the numerical diffusivity for the case of X = 0. From Eq.(12), the grid diffusivity is

$$\nu_g = \frac{c\Delta x}{2} \qquad \text{....(16)}$$

The cell Reynold number (Roache, P., 1972) is defined as the ratio of hydraulic diffusivity (Eq.13) to grid diffusivity (Eq.16). This leads to

$$D = \frac{q_o}{S_o c\Delta x} \qquad \text{....(17)}$$

in which, D = Cell Reynolds number. Therefore, from Eq.(14) and (17)

$$X = \frac{1}{2}(1-D) \qquad \text{....(18)}$$

Eq.(17) and (18) imply that for very small values of Δx, D may be greater than 1, leading to negative values of X. In fact, for the characteristic reach length

$$\Delta x_c = \frac{q_o}{S_o c} \qquad \text{....(19)}$$

the cell Reynold number is D = 1, and X = 0. Therefore, in the Muskingum-Cunge method, reach length shorter than the characteristic reach length result in negative values of X. This should be contrasted with the classical Muskingum method, in which X is restricted in the range 0.0 - 0.5. In the classical Muskingum, X is interpreted as a weighting factor. As shown by Eq.(17), and (18) non negative values of X are associated with long reaches (Δx more than characteristic length Δx_c given by Eq.(19), typical of the manual computation used in the development and early application of the Muskingum method.

In the Muskingum-Cunge method, however, X is interpreted in a moment matching sense (USDA, 1973) or diffusion matching factor. Therefore, negative values of X are entirely possible. This feature allows the use of shorter reaches than would otherwise be possible if X were restricted to non-negative values.

The substitution of Eq.(15) and (18) into Eq.(8) to (10) leads to routing co-efficients expressed in terms of courant and cell Reynolds numbers:

$$C_o = \frac{-1 + C + D}{1 + C + D} \quad \ldots.(20)$$

$$C_1 = \frac{-1 + C + D}{1 + C + D} \quad \ldots.(21)$$

$$C_2 = \frac{-1 + C + D}{1 + C + D} \quad \ldots.(22)$$

Thus C and D are the two routing parameters required to be estimated for Muskingum-Cunge method.

2.1 Estimation of Routing Parameters

(a) Estimation of parameter C (Courant number)

The parameter C can be estimated using Eq.(15). It requires an estimate for wave celerity (c) in addition to grid size (Δx, Δt). The wave celerity can be calculated with either

$$c = \beta V \quad \ldots.(23)$$

or

$$c = \frac{1}{T}\frac{dQ}{dy} \quad \ldots.(24)$$

Where, β is an exponent in the discharge area rating equation given as

$$Q = \alpha (A)^{\beta} \quad \ldots.(25)$$

The calculation of β is a function of frictional type and cross sectional shape.

Theoretically Eq.(23) and (24) are the same. For practical applications, if a stage-discharge rating and cross sectional geometry are available (i.e. stage - discharge - top width tables), Eq.(24) is preferred over Eq.(23) because it accounts directly for cross sectional shape. In the absence of a stage - discharge rating and cross sectional data, Eq.(23) can be used to estimate flood wave celerity. The velocity V in Eq.(23) can be taken as the velocity at reference flow. The choice of reference flow has bearing on the calculated results although the overall effect is likely to be small. The peak flow value has the advantage that it can be readily ascertained, although a better approximation may be obtained by using an average value.

(b) **Estimation of parameter D (Cell Reynold number)**

Cell Reynold numbers (D) can be calculated using the reach length (Δx), reference discharge per unit width q_o, kinematic wave celerity (c), and bottom slope (S_o) in Eq.(17).

2.2 Resolution Requirements

When using the Muskingum-Cunge method sufficiently small values of Δx and Δt should be taken in order to approximate closely the actual shape of the hydrograph. For smoothly rising hydrographs, a minimum value of $t_p/\Delta t = 5$ is recommended. This requirement usually results in the hydrograph time base being resolved into atleast 15 to 25 discrete points, considered adequate for Muskingum routing.

Unlike temporal resolution, there is no definite criteria for spatial resolution. A criterion borne out by experience is based on the fact that courant and Cell Reynolds numbers are inversely related to reach length Δx. Therefore, to keep Δx sufficiently small, courant and Cell Reynolds numbers should be kept sufficiently large. Thus leads to the practical criterion:

$$C + D \geq 1 \qquad \text{....(26)}$$

Which can be writtern as: $-1 + C + D \geq 0$.

This confirms the necessity of avoiding negative values of C_o in Muskingum-Cunge routing (Eq.7). Experience has shown that negative values of either C_1 or C_2 do not adversely effect the methods over all accuracy (Ponce and Theurer, 1982).

Notwithstanding Eq.(26), the Muskingum Cunge method works best when the numerical dispersion is minimised, that is, when C is kept close to 1. Values of C sufficiently different from one are likely to cause the notorious dips, or negative outflows, in portions of the calculated hydrograph. This computational anomaly is attributed to excessive numerical dispersion and should be avoided.

3.0 METHODOLOGY

The steps involved in flood routing through a channel reach using the Muskingum-Cunge Method are given as follows:

(i) Estimate the parameter C (Courant number) using the following equation:

$$C = c\frac{\Delta t}{\Delta x}$$

The wave celerity c is computed using the procedure described in section 2.1. The temporal and spatial resolutions (Δt and Δx) should be such that the routing co-efficient C_o should not be negative as well as the value of courant number (C) should be close to one in order to minimise the numerical dispersion.

(ii) Estimate the parameter D (Cell Reynolds number) using the following equation

$$D = \frac{q_o}{S_o C\Delta x}$$

The wave celerity (c) and reference discharge, $q_o (=Q_o/T)$ per unit width are used togather with channel slope S_o and reach length Δx in the above equation to provide the parameter D (Cell Reynolds number). Please ensure whether $-1 + C + D \geq 0$, which is the practical criterion to avoid the negative values of C_o in Muskingum Cunge routing.

(iii) Estimate the routing co-efficients C_o, C_1 and C_2 using the following equation:

$$C_o = \frac{-1 + C + D}{1 + C + D}$$

$$C_1 = \frac{1 + C - D}{1 + C + D}$$

$$C_2 = \frac{1 - C + D}{1 + C + D}$$

(iv) Route the inflow hydrograph (Q) using the following equation in order to have the outflow hydrograph (Q_{j+1}):

$$Q_{j+1}^{n+1} = C_o\, Q_j^{n+} + C_1\, Q_j^n + C_2\, Q_{j+1}^n$$

(v) If the channel is divided into sub-reaches, the steps (i) to (iv) should be repeated for all the sub-reaches considering the outflow from the first sub-reach as inflow to second sub-reach and so on.

Example 1: Use the Muskingum-Cunge method to route a flood wave with the following flood and channel characteristics:

peak flow Q_p = 1000 m^3/s

Base flow Q_b = 0 m^3/s

Channel bottom slope S_o = 0.000868

flow area at peak discharge A_p = 400 m^2,

top width at peak discharge T_p = 100 m

rating exponent β = 1.6,

reach length Δx = 14.4 Km.

time interval Δt = 1 hr.

Time (h)	0	1	2	3	4	5	6	7	8	9	10
Flow (m^3/s)	0	200	400	600	800	1000	800	600	400	200	0

Solution:

(i) Compute the wave clerity (c)

$c = \beta V$

here β = 1.6 and the mean velocity (based on the peak discharge) is $V = Q_p/A_p = 1000/400 = 2.5$ m/s

$$c = 1.6 \times 2.5 = 4 \text{ m/s}$$

(ii) Compute the Courant number (C)

$$C = c\frac{\Delta t}{\Delta x} = \frac{4 \times 1 \times 3600\ 10^{-3}}{14.4} = \frac{14.4}{14.4} = 1.0$$

(iii) Compute the Cell Reynold number (D)

$$D = \frac{q_o}{S_o c \Delta x}$$

here q_o = the flow per unit width (based on the peak discharge)

$$= \frac{Q_p}{T_p} = \frac{1000}{100} = 10 \text{ m}^2/\text{s}$$

$$D = \frac{10}{0.000868 \times 4 \times 14.4 \times 10^3} = \frac{10}{0.368 \times 57.6}$$

$$= 0.2$$

(iv) Compute the routing co-efficients

$$C_o = \frac{-1 + C + D}{1 + C + D} = \frac{-1 + 1 + 0.2}{1 + 1 + 0.2} = 0.091$$

$$C_1 = \frac{1 + C - D}{1 + C + D} = \frac{1 + 1 - 0.2}{1 + 1 + 0.2} = 0.818$$

$$C_2 = \frac{1 - C + D}{1 + C + D} = \frac{1 - 1 + 0.2}{1 + 1 + 0.2} = 0.091$$

(v) Compute the outflow hydrograph using the following routing equation:

$$Q_{j+1}^{n+1} = C_o Q_j^{n+1} + C_1 Q_j^n + C_2 Q_{j+1}^n$$

$$= 0.091\ Q_j^{n+1} + 0.818\ Q_j^n + 0.091\ Q_{j+1}^n$$

The routing calculations are shown in Table 2

Table 2 : Channel Routing by Muskingum-Cunge Method

Time	Inflow	Partial Flows			Outflow
	(Q_j^{n+1})	$C_o Q_j^{n+1}$	$C_1 Q_j^n$	$C_2 Q_{j+1}^n$	(Q_{j+1}^{n+1})
(hr)	(m^3/s)	(m^3/s)	(m^3/s)	(m^3/s)	(m^3/s)
(1)	(2)	(3)	(4)	(5)	(6) = (3)+(4)+(5)
0	0	-	-	-	0
1	200	18.2	0	0	18.2
2	400	36.4	163.6	1.66	201.66
3	600	54.6	327.2	18.35	400.15
4	800	72.8	490.8	36.41	600.01
5	1000	91.0	654.4	54.60	800.00
6	800	72.8	818.0	72.80	963.60
7	600	54.6	654.4	87.69	796.69
8	400	36.4	490.8	72.50	599.70
9	200	18.2	327.2	54.57	399.97
10	0	0	163.6	36.40	200.00
11	0	0	0	18.20	18.20
12	0	0	0	1.66	1.66
13	0	0	0	0.16	0.16

4.0 NON LINEAR MUSKINGUM-CUNGE METHOD

The kinematic wave-equation (Eq.5) is non linear because the kinematic wave celerity varies with discharge. The non linearity is mild because the wave celerity variation is usually restricted within a narrow range. However, it becomes necessary to account this non linearity in certain cases. This can be achieved in two ways: (i) during the discretization, by allowing the wave celerity to vary, resulting in a non linear numerical scheme to be solved by iterative means; and (ii) after the discretization by varying the routing parameters, as in the variable parameter Muskingum-Cunge Method where the routing parameters are allowed to vary with the flow. The latter approach is particularly useful if the overall non-linear effect is small, which is often the case.

In the variable Parameter Muskingum-Cunge method the values of C and D are based on local q_o and c values instead of peak flow or other reference value in the constant - parameter method. To vary the routing parameters, the most expedient way is to obtain an average of q_o and c for each computational cell. This can be achieved with a direct three point average of the values at the known grid points (Fig.1), or by an iterative four point average, which includes the unknown grid point. To improve the convergence of iterative four point procedure, the three point average can be used as the first guess of the iteration. Once q_o and c have been determined for each computational cell, the courant and cell Reynolds numbers are calculated by Eq.(15) and (17). The value of bottom slope, S_o, remains unchanged within each computational cell.

The variable parameter Muskingum-Cunge method represents a small yet some times perceptible improvement over the constant parameter method. The differences are likely to be more marked for very long reaches and/or wide variations in flow levels. Flood Hydrographs calculated with variable parameters show a certain amount of distortion, either wave steepening in the case of flows contained in bank or wave attenuation in the case of typical over bank flows. This is a physical manifestation of the non-linear effect i.e. different flow levels travelling with different celerities. On the other hand, flow hydrographs calculated using constant parameters do not show such wave distortion.

5.0 ADVANTAGES AND LIMITATIONS OF MUSKINGUM-CUNGE METHOD

(i) The Muskingum-Cunge method is a physically based alternative to the Muskingum method. Unlike Muskingum method where the parameters are calibrated using streamflow data, in the Muskingum Cunge Method the parameters are calculated based on flow and channel characteristics. This makes possible channel routing without the need for time consuming and cumber some parameter calibration. More importantly, it makes possible extensive channel routing in ungauged streams with a reasonable expectation of accuracy.

(ii) With the variable parameter feature, non-linear properties of flood waves (which could otherwise only be obtained by more elaborate numerical procedures) can be described within the context of the Muskingum formulation.

(iii) Lithe Muskingum method, the Muskingum-Cunge method is limited to difussion waves. Furthermore, Muskingum-Cunge method is based on a single valued rating and does not take into account strong flow non uniformity or unsteady flows exhibiting substantial loops in dischrage-stage rating (i.e. dynamic waves). Thus the Muskingum-Cunge method is suited for channel routing in natural streams without significant back water effects and for unsteady flows that classify under the diffusion wave criterion.

(iv) An important difference between the Muskingum and Muskingum-Cunge methods is that the former is based on the storage concept and, therefore, the parameters K and X are reach averages. The later method, however, is kinematic in nature, with the parameters C and D being based on values evaluated at channel cross sections rather than being reach average. Therefore, for the Muskingum-Cunge method to improve on the Muskingum method, it is necessary that the routing parameters evaluated at channel cross sections be representative of the channel reach under consideration.

(v) Historically, the Muskingum Method has been calibrated using stream flow data. On the contrary, the Muskingum-Cunge method relies on physical characteristics such as rating curves, cross sectional data and channel slope. The different data requirements reflect the different theoretical bases of the methods i.e. storage concept in the Muskingum method, and kinematic wave theory in the Muskingum-Cunge method.

6.0 REMARKS

Muskingum-Cunge method is a better alternative to the Muskingum method. It works best when the numerical dispersion is minimised. The method can be applied to the ungauged reach without any difficulty as it involves the use of physical characteristics for the estimation of parameters. Since the method is limited to diffusion waves and does not take into account the substantial loops in discharge-stage rating (i.e. dynamic waves), its application is restricted for channel routing of the natural streams without significant backwater effects and for unsteady flows that classify under the diffusion wave criterion.

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