LECTURE-7

# HYDRAULIC METHODS OF FLOOD ROUTING THROUGH CHANNELS

by

Dr. S K Jain,

Scientist E, NIH, Roorkee.

## INTRODUCTION

The term channel routing can be defined as the mathematical method to predict the changing magnitude, shape & speed of a flood wave as it propagates through waterways such as canals, rivers, reservoirs or estuaries, Fread (1985). The channel routing methods can be classified in three categories : empirical methods, hydrologic methods, and hydraulic methods.

Since a flood is a unsteady flow phenomenon, it can be best described by equations which take into account the unsteady aspect of flow. The basic equation of unsteady flow were first developed by a french scientist DE SAINT VENANT in 1871 and are called the Saint Venant equations. The implicit method of routing which is concerned with solution of the St. Venant equations is described in the following.

## SAINT VENANT EQUATIONS

These equations are the fundamental equations of unsteady flow. Although several modifications of these equations have been suggested since their development, the basic equations have remained the same. These equations are :

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} - Q_s = 0 \qquad (1)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial}\left(\frac{Q^2}{A}\right) + Q_s A v = gA\left(S_0 - S_f - \frac{\partial y}{\partial x}\right) \qquad (2)$$

where,

A = wetted cross section area of channel,

Q = discharge in the channel,

$Q_s$ = lateral inflow per unit channel length,

g = acceleration due to gravity,

y = depth of flow,

x = distance along the channel from some upstream reference point,

$S_o$ = bottom slope of the channel,

$S_f$ = friction slope,

v = velocity of lateral inflow in the direction of main flow, and

t = time.

Of the above equations, eq. (1) has been derived from continuity principle and eq. (2) has been derived from momentum principles. In the equation (2), the first two terms represent the rate of change of momentum of water, the third and fourth term represent change in pressure force and the last term is due to resistance effects. These two equations form a set of non-linear first order hyperbolic partial differential equations describing unsteady flow in open channels.

The fundamental assumptions underlying the derivation of Saint Venant equations are :

1) The water surface varies gradually. This implicates that the pressure distribution along a vertical is hydrostatic and vertical acceleration is small.

2) The friction losses in unsteady flow are not significantly different from those in steady flow.

3) Velocity distribution along the wetted area does not substantially affect the wave propagation. In other words, the velocity distribution is uniform.

4) The bottom slope of the channel is small so that
$\sin\theta \approx \tan\theta \approx \theta$
$\cos\theta \approx 1$.

5) The channel is straight and prismatic.

Attempts have been made to establish the experimental verification of St. Venant equations. It has been found that the results obtained by numerical integration of these equations are sufficiently close to the experimental results.

NUMERICAL SOLUTION OF ST. VENANT EQUATIONS

It is not possible to obtain closed form analytical solution of St. Venant equations. In the beginning of current century, some graphical methods were developed and were in use for quite some time. But with the advent of fast computers numerical methods have taken a lead over them. Now with the help of computers, solutions up to the desired accuracy can be obtained quickly.

Due to various problems associated with the solution of complete St. Venant equations, various simplifications have been suggested by various investigators. These simplifications are classified as the kinematic wave model and the diffusive wave model. These are depicted in the following :

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{3}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial}\left(\frac{Q^2}{A}\right) + gA\frac{\partial y}{\partial x} + gA(S_f - S_0) = 0 \tag{4}$$

- $gA(S_f - S_0)$ — Kinematic Wave
- $gA\frac{\partial y}{\partial x} + gA(S_f - S_0)$ — Diffusive Wave
- All terms — Dynamic Wave

The available numerical methods of solution of St. Venant equations can be classified as :

(a) Method of characteristics,
(b) Explicit finite difference methods,
(c) Implicit finite difference methods, and
(d) Finite element methods.

In the following, the implict methods are described in details.

IMPLICIT METHODS OF FLOOD ROUTING

Since all real life floods have to be routed for a relatively long duration and the distances involved are also comparatively large, the earlier discussed methods were found unsuitable. The need and search for a method which could work under such conditions led to the development of implicit routing techniques.

In the implicit methods, the x derivatives of the unknown terms of Saint Venant equations are replaced in terms of the finite differences evaluated at time $t_o$+t. The resulting equations are nonlinear in character.Since the unknowns implicitly appear in the equations, the name of solution method is implicit method. The solution of these equations is more complex than in the explicit method.

Although the various implicit schemes are unconditionally linearly stable, instability could occur if the time step size is too large and the x-derivative terms are not sufficiently advanced towards the future time line when modelling rapidly varying transients. Also a too large time step size should not be used for the sake of accuracy. In general, $\Delta t$ depends on the $\Delta x$ size, the shape of the hydrograph, the type of implicit scheme chosen and the nonlinearities introduced due to irregularities of the cross-section etc.

In past several implicit schemes have been developed. The first detailed description of an implicit scheme appeared in 1956. One of the earliest developed implicit scheme for flood routing was suggested by Preissmann (1961). He adopted a four point grid in (x,t) plane and also introduced a weighting factor. Liggett and Woolhiser (1967)developed a six point implicit scheme. The method permitted slightly larger values of time steps than that allowed in characteristics method but it is difficult to apply a six point method to an irregular space mesh. However Abbott and Ionescu (1967) successfully applied the six point scheme to irregular space mesh by using a weighting procedure.

Amein and Fang(1970) used a four point scheme or box scheme and an improved versions of this scheme was used by Fread(1973) and Amein and Chu (1975). This scheme has also been used in the Dynamic Wave Model, FLDWAV, developed by Fread(1985).

FOUR POINT IMPLICIT SCHEME

This scheme is also termed as Preissmann scheme. In this scheme the numerical solution of equations (1) and (2) is obtained in two steps. In the first step the equations are replaced by two algebraic finite difference equations and the second step consists of solution of these equations. The salient features of the Preissmann type schemes are the following :

- They are consistently approximate integral conservation laws,
- They compute both unknown variables at the same computational grid points,
- They link together flow variables at only two adjacent sections; thus the space intervals may be variable while the accuracy of approximation is unaffected,
- They are schemes of first order approximations except for the special case when $\theta$=0.5 when the approximation is of the second order,

- For a special choice of $\Delta t$ and $\Delta x$, they furnish the exact solution for the fully linearized flow equations.

The numerical solution is obtained over a discrete rectangular net of points on (x,t) plane as shown in Fig. 1. The lines drawn parallel to x-axis represent time and those parallel to t-axis represent a channel location. The intersection points of these lines are called node points. The spacing of these lines is $\Delta t$ and $\Delta x$ respectively which need not be constant. The scheme is termed as four point scheme because the continuity and momentum equations are applied to two adjacent computational cross sections.

Let $\alpha$ be any variable such as Q, y, A or u etc. The partial derivatives of with respect to t and x can be expressed as :

$$\partial\alpha(M)/\partial t = [\ \alpha(P) + \alpha(T) - \alpha(R) - \alpha(S)\ ]/2\Delta t \qquad (5)$$

$$\partial\alpha(M)/\partial x = (1-\theta)[\alpha(R) - \alpha(S)]/\Delta X + \theta[\alpha(P)-\alpha(T)]/\Delta x \qquad (6)$$

Where
$\theta$ = a weighting factor, $\Delta t'/\Delta t$

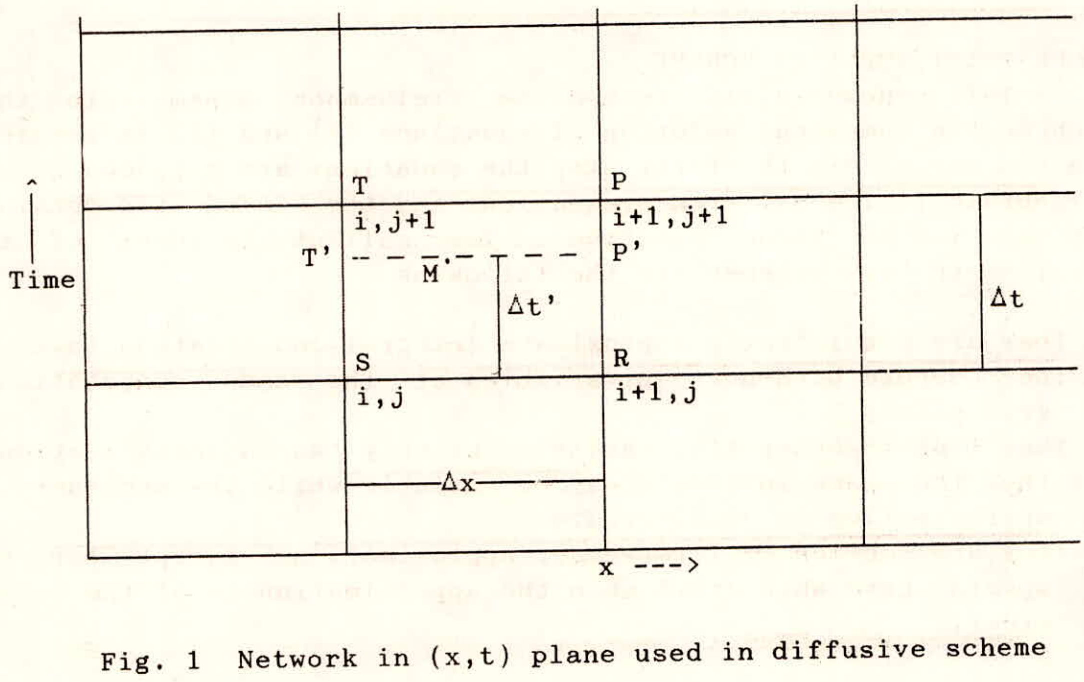


Fig. 1 Network in (x,t) plane used in diffusive scheme

The factor $\theta$ has been introduced to impart artificial damping in the system. The value of $\theta$ varies from 0 to 1. When $\theta=0$, the scheme reduces to explicit method. The value 0.5 produces box scheme or centered finite difference scheme. The weighting factor allows flexibility in placing derivative terms. The weighting factor must greater than 0.5 to provide unconditional linear stability with respect to time step size. Fread(1985) has pointed out that the accuracy of the scheme generally decreases as the weighting factor approaches unity. The effect becomes more pronounced with increase in the time step size. However, the resulting equations with $\theta$ =1are simpler in structure. While the box scheme is stable for slowly varying flow, it may produce numerical oscillations under certain conditions. A value of $\theta=0.55$ has been recommended to minimize the loss of accuracy and to avoid weak instability. To generalize the approach, the equations given here are with $0.5 \leq \theta \leq 1$.

Using equations (5) and (6), the equation of continuity or equation (1) can be expressed in the finite difference form as :

$$[(1-\theta)(Q_{i+1}^{j}-Q_{i}^{j}) + \theta(Q_{i+1}^{j+1}- Q_{i}^{j+1})]/(x_{i+1}- x_{i}) +0.5[(A_{i+1}^{j+1} + A_{i}^{j+1}) - (A_{i+1}^{j}+ A_{i}^{j})]/(t^{j+1}- t^{j}) - Q_{s}(x,t) = 0 \qquad (7)$$

Here subscripts refer to longitudinal location and superscripts refer to time step. Simplifying the above equation :

$$[(1-\theta)(Q_{i+1}^{j}-Q_{i}^{j}) + \theta(Q_{i+1}^{j+1}- Q_{i}^{j+1})] +0.5\Delta x/\Delta t[(A_{i+1}^{j+1} + A_{i}^{j+1} - A_{i+1}^{j}- A_{i}^{j})] - Q_{s}(x,t)\Delta x = 0 \qquad (8)$$

Similarly the finite difference form of momentum equation or equation (2) is obtained as :

$$0.5[(Q_{i+1}^{j+1} + Q_i^{j+1}) - (Q_{i+1}^{j} + Q_i^{j})]/(t^{j+1} - t^{j}) + [(1-\theta)\{(Q_{i+1}^{j})^2/A_{i+1}^{j} - (Q_i^{j})^2/A_i^{j}\} + \theta\{(Q_{i+1}^{j+1})^2/A_{i+1}^{j+1} - (Q_i^{j+1})^2/A_i^{j+1}\}]/(x_{i+1} - x_i) - 0.5g[\theta(A_{i+1}^{j+1} + A_i^{j+1}) + (1-\theta)(A_{i+1}^{j} + A_i^{j})]\;[\{(1-\theta)(y_{i+1}^{j} - y_i^{j}) + \theta(y_{i+1}^{j+1} - y_i^{j+1})\}/(x_{i+1} - x_i) + 0.5\{(1-\theta)(S_{f_{i+1}}^{j} + S_{f_i}^{j}) + \theta(S_{f_{i+1}}^{j+1} + S_{f_i}^{j+1})\} - 2S_o] + Q_s(x,t)V(x,t) = 0 \tag{9}$$

or

$$0.5[Q_{i+1}^{j+1} + Q_i^{j+1} - Q_{i+1}^{j} - Q_i^{j}] + \Delta t/\Delta x\;[(1-\theta)\{(Q_{i+1}^{j})^2/A_{i+1}^{j} - (Q_i^{j})^2/A_i^{j}\} + \theta\{(Q_{i+1}^{j+1})^2/A_{i+1}^{j+1} - (Q_i^{j+1})^2/A_i^{j+1}\}] - 0.5g\Delta t[\theta(A_{i+1}^{j+1} + A_i^{j+1}) + (1-\theta)(A_{i+1}^{j} + A_i^{j})]\;[\{(1-\theta)(y_{i+1}^{j} - y_i^{j}) + \theta(y_{i+1}^{j+1} - y_i^{j+1})\} + 0.5\{(1-\theta)(S_{f_{i+1}}^{j} + S_{f_i}^{j}) + \theta(S_{f_{i+1}}^{j+1} + S_{f_i}^{j+1})\} - 2S_o] + Q_s(x,t)V(x,t) = 0 \tag{10}$$

The equations (8) and (10) are finite difference form of equations (1) and (2) as used in the implicit scheme. All the variables occurring with superscript j are known and with superscript j+1 are unknown. However, all the unknowns are not independent. Thus equations (8) and (10) contain only four unknowns, i.e., the value of discharge and depth at grid points (i,j+1) and (i+1,j+1). Another striking feature is that $\Delta x$ and $\Delta t$ need not be constant and $\Delta x$ can be varied at any $x_i$ and $\Delta t$ can be varied at any time $t_j$.

These two equations constitute a system of two nonlinear algebraic equations in four unknowns. If a channel has N sections or (N-1) grids than in all 2N unknowns need to be evaluated at any state of iteration. Equations (8) and (10) can be written for any grid. But since two unknowns are common to two adjacent grids, the application of the equations gives 2(N-1) equations.

Hence two additional equations are needed to solve this system of equations. They are provided by boundary conditions.

BOUNDARY CONDITIONS

The two boundary conditions required to solve the system of equations provided by the upstream and downstream boundaries.

The upstream boundary condition can :

(a) Flow depth as a function of time, or
(b) Discharge as a function of time.

The downstream boundary condition can be :

(a) The flow depth as a function of time, or
(b) Discharge as a function of time, or
(c) A stage discharge relationship - say as rating curve or the relationship if a control structure such as a weir or gate etc. exists.

If the depth at upstream boundary is known as a function of time then :

$$y_1^{j+1} - y'(t^{j+1}) = 0 \qquad (11)$$

where $y'(t^{j+1})$ = known depth of flow at upstream boundary at time $t^{j+1}$.

If discharge is known instead of depth then :

$$Q_1^{j+1} - Q^1(t^{j+1}) = 0 \qquad (12)$$

Either of equations (14) and (15) can be used as a supplementary equation. In generalized form we can write :

$$F_o(y_1, Q_1) = 0$$

Similarly at downstream boundary, if the depth is known as a function of time then :

$$y_N^{j+1} - y_N''(t^{j+1}) = 0 \quad (13)$$

If discharge is known as a function of time then :

$$Q_N^{j+1} - Q_N''(t^{j+1}) = 0 \quad (14)$$

Again equation (13) or (14) can be used as second supplementary equation. In general form :

$$F_N(y_N, Q_N) = 0$$

SOLUTION OF SIMULTANEOUS EQUATIONS

Putting together all the equations, 2N simultaneous equations for 2N unknowns are obtained. These are assembled here :

$$\begin{aligned}
F_0(y_1,\ Q_1) &= 0 \\
F_1(y_1,\ Q_1,\ y_2,\ Q_2) &= 0 \\
G_1(y_1,\ Q_1,\ y_2,\ Q_2) &= 0 \\
F_i(y_i,\ Q_i,\ y_{i+1}, Q_{i+1}) &= 0 \\
G_i(y_i,\ Q_i,\ y_{i+1}, Q_{i+1}) &= 0 \\
F_N(y_N,\ Q_N) &= 0
\end{aligned} \quad (16)$$

Here symbol $F_i$ denotes equations obtained by application of equation (8) and $G_i$ for those obtained from equation (10).

Equations (16) form a system of 2N equations in 2N unknowns. Any eliminations or matrix inversion can be used to solve them. Each time the system is solved, a better approximation of unknowns is obtained. The process can be stopped when the required accuracy is obtained. The computations can be now advanced to the for next time step. As for initial approximation, the value of unknowns at previous time step can be taken as starting value. It has been reported that instead of linear interpolation a parabolic interpolation gives faster convergence.

The Newton-Raphson method, which makes use of Taylor series expansion of a nonlinear function by neglecting all terms of second and higher order, is an iterative technique to solve a system of nonlinear equations. The algorithm is :

$$J'(X^k)\Delta X = -f(X^k) \tag{17}$$

where, $X^k$ is a vector, J' is the Jacobian, and $f(X^k)$ is the nonlinear equation evaluated with $X^k$ values, and $\Delta X$ is a vector containing 2N unknowns in terms of flow depth and discharge. The Jacobian matrix is composed of the partial derivatives. The vector $\Delta X$ represents the difference between the solution at two successive stages of iteration, i.e.,

$$\Delta X = X^{k+1} - X^k \tag{18}$$

where k is the iteration number. Convergence is achieved when the quantities contained in $\Delta X$ become less than some specified value. The final values of depth and discharge at the previous time interval provide a good starting estimate of the unknowns. Fread(1985) suggests following formula to obtain the first estimates :

$$X^k = X^{j-1} + (X^{j-1} - X^{j-2})\alpha \tag{19}$$

where $\alpha$ is a weighting factor in the range zero to one, $X^j$ is the solution vector at time j.

The coefficients of equations (16) are banded around main diagonal and some efficient solution technique taking advantage of the banded structure of the coefficient matrix can be used for fast solution. Such procedures have been suggested by Fread(1971) and Liggett and Cunge(1975).

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